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DYNAMICS OF TURBULENT FLOWS

Paolo Mele
e Michele
La Rocca

Roma TrE-Press
2023

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Collana
QUADERNI DI INGEGNERIA
Dipartimento di Ingegneria Civile, Informatica e delle Tecnologie Aeronautiche - DICITA -
Direttore della collana: Michele La Rocca

Coordinamento editoriale
Gruppo di Lavoro *Roma TrE-Press*

Cura editoriale e impaginazione
teseo editore Roma teseoeditore.it

Elaborazione grafica della copertina
MOSQUITO. mosquitoroma.it

Edizioni: *Roma TrE-Press* ©
Roma, aprile 2023
ISBN: 979-12-5977-161-2

http://romatrepress.uniroma3.it

L'attività della *Roma TrE-Press* è svolta nell'ambito della
Fondazione Roma Tre-Education, piazza della Repubblica 10, 00185 Roma.

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CHAPTER 1 INTRODUCTION

The first qualitative descriptions of turbulent flows were made by Leonardo da Vinci (1452-1519), whose drawings sometimes illustrate turbulent phenomena observed in natural watercourses, but the first scientific approach to the study of turbulence is due to Osborne Reynolds (1842-1912), who showed the importance of the dimensionless number, known as the Reynolds number, and introduced the concept of turbulent stress. Most subsequent studies, even the most recent ones are based on the approach he followed. It consists in representing all the hydrodynamic variables of a turbulent flow as the sum of two components in the framework of a statistical approach: the average and the fluctuating or turbulent component. Applying appropriate operations to the equations of motion, Osborne Reynolds obtained the so-called Reynolds equations that describe the dynamics in terms of average components. The non-linearity of the equations of motion, however, gives raise to the interaction between the fluctuating and the average components through the turbulent stresses or Reynolds stresses. It is also worth mentioning Lewis Richardson's work (1922-1926). He introduced only general ideas in qualitative form without making any deduction that could be formulated in a precise mathematical language but he gave a thoughtful insight into the dynamics of turbulence. To him we owe the concept that the developed turbulence consists of a hierarchy of "eddies" of various magnitude. The word "eddy", still used today, was not defined exactly[1] but was used by him roughly with the meaning of ***vortex***, i.e. as a flow tube with a vorticity concentrated inside and negligible outside, and also in a sense of disturbance or non-homogeneity of the motion. The process of energy cascade, later formulated in a clear and accurate way by other scientists, was synthesized by

[1] The vorticity, on the contrary, is well defined and is crucial in the turbulence dynamics. Note that there is an important distinction between vorticity and vortices: for example, a laminar boundary layer has vorticity but no vortices.

Richardson in the following verses:

Big whorls have little whorls,
Which feed on their velocity;
And little whorls have lesser whorls
And so on to viscosity (in the molecular sense).

To Geoffrey Taylor (1886-1975), Ludwig Prandtl (1875-1953) and his pupil Theodor von Karman (1881-1963) is due the semi-empirical approach to the theory of turbulence. The semi-empirical theories originate from the analogy between turbulence and molecular disorder. These theories do not form a single basis and are rather based on simple assumptions of analogy or similarity, and on experimental results, than on theoretical models of the dynamics of turbulence. The original fundamental concepts included mixing length, turbulent intensity, and turbulent (eddy) viscosity coefficients. The results are immediately usable, both in engineering flows, such as in pipes, open channels and in boundary layers, and in flows of theoretical interest such as in free turbulence, i.e. in absence of fixed impermeable walls. The possibilities of the semi empirical method are not yet exhausted and useful work in this direction is still developing at the present days.
Taylor also gave a fundamental conceptual contribution, postulating the probabilistic nature of turbulence, considering the turbulent hydrodynamic variables as random variables for which probability density functions should be determined. He introduced the investigation, developed by his disciples Batchelor (1920-2000) and Townsend (1917-2010), on isotropic and homogeneous turbulence. In thirty years (1930-1960), was developed the statistical formulation, which obtained interesting results, but at the same time showed the difficulties of framing the turbulent phenomenon with simple theories, even using techniques other than those commonly used in Fluid Mechanics. The state of knowledge is well represented by Batchelor, Townsend, Hinze (1907-1993) and, with a clearer and more precise approach, by Monin (1921-2007) and Yaglom (1921-1988). The greatest contribution in that period was

given by A.N. Kolmogorov (1903-1987) on the local structure of turbulence. In the early 60s, first Kraichnan (1928-2008) and then Edwards (1928-2015) applied the field theory techniques of Quantum Mechanics. The results, however, were, and still are, purely formal and of little use for the concrete resolution of engineering problems. The formal analogy of the field theory of Quantum Mechanics had been, on the other hand, already present in Landau's (1908-1968) approach.

Thus at the end of the 1960s it was commonly accepted that the complexity of turbulent motion was due to the excitation of a large number of degrees of freedom and therefore it could be and should be expressed by a large number of differential equations. On the contrary, in the laminar motion the excited degrees of freedom would be very few. We can describe the motion of a continuous medium by means of an infinite number of generalized coordinates, i.e. by means of the coefficients of the expansion of the velocity field with respect to a complete system of functions. In other words, the velocity vector field can be expressed as a linear combination of assigned, linearly independent, velocity vector fields. The coefficients of the combination represent the coordinates relative to the representation basis. The coefficients can assume different values with time, generating trajectories in the representation space. A possible system of ordinary differential equations, which represents a model of time evolution, must have a number of degrees of freedom, i.e. a number of differential equations, equal to the dimensions of the representation space. For a laminar flow, almost all the coefficients would be negligible, only very few would not be negligible and the corresponding degrees of freedom excited. For a turbulent flow a great number of degrees of freedom would be excited and the resulting motion would be extremely complex.

Nevertheless, in the early 1960s, the American meteorologist Lorenz (1917-2008) showed that even simple systems of nonlinear ordinary differential equations may have solutions with a chaotic behaviour, not due to the influence of a large number of non-controllable factors, but due to the nature of the system of equations. This observation paved the way for the

chaos theory approach to turbulence, according to which motions with very close initial conditions evolve in a completely different way. Ruelle (1935-) and Takens (1940-2010) assumed that this is what happens in turbulent flows, introducing new attractors, called ***strange attractors***, of the solutions of nonlinear dynamical systems.

The chaos theory approach to turbulence has been developed in order to obtain the asymptotic behaviour of solutions of nonlinear differential equations systems (dynamic systems) and to investigate the existence of discontinuous solutions (bifurcation theory). With this regard, the availability of computational resources is a key factor and in the last decades many important results in specific applications have been obtained thanks to the tremendous development of computational resources and methods. However, the turbulent mechanism as a whole is still wrapped up in mystery, even though concrete results have been achieved in specific applications. The methodological tools used to obtain results in the study of turbulent motion are based on dimensional analysis, experimental prototypes and numerical models. More recently, important developments have been obtained thanks to Computational Fluid Dynamics (CFD).

From the fall of water described by Leonardo da Vinci and from the first scientific experiments of Reynolds until today, turbulence continues to be the subject of study and research. Turbulence is still an open problem in many respects and research on it is continuously in progress.

Figure 1.1 Leonardo da Vinci: Water passing obstacles and falling.

Source: https://it.m.wikipedia.org/wiki/File:Studies_of_Water_passing_Obstacles_and_falling.jpg

CHAPTER 2 HYDRODYNAMIC STABILITY

2.1. Instability phenomena

If a steady, uniform flow impinges on a flat plate, the boundary layer is steady too in the neighborhoods of the leading edge of the plate, but moving from the leading edge, the flow becomes gradually unsteady. In other words, at a given distance from the leading edge the flow quantities (velocity, pressure, etc.) oscillate with respect to both time and space. The oscillations of the flow quantities become more and more complex increasing the distance from the leading edge of the plate, until they appear completely chaotic and the flow is considered turbulent. It is evident that the steady boundary layer solution, obtained from the Navier-Stokes equation, does not represent the flow behavior at all.
This behavior can be seen in the Poiseuille flow too: the well-known parabolic velocity profile in the pipe is steady until the dimensionless parameter:

$$Re = \frac{U_0\, h}{\nu} \tag{2.1}$$

is small enough. In equation 2.1, U_0 , h and ν are the flow velocity-scale (e.g. the cross section average velocity of the steady velocity profile), the flow characteristic length-scale (e.g. the diameter of the pipe) and the kinematic viscosity of the fluid. On the contrary, as soon as the dimensionless parameter, known as Reynolds number in honor of Osborne Reynolds, is large enough, time and space fluctuations affect the flow quantities. It is important to observe that these fluctuations occur only if the Reynolds number is large enough, but the way with which these fluctuations affect the flow depends on the value of the Reynolds number. The steady flow in a pipe is always a solution of the steady Navier-Stokes equation, whatever the Reynolds number, but it is unstable for high values of the

Reynolds number. This means that even small perturbations are amplified, instead of being damped out and so the actual flow evolves in a different way, with respect to that foreseen by the steady solution.

Figure 2.1: A capillary jet breaking into droplets (Rayleigh-Plateau Instability). Source: https://commons.wikimedia.org/wiki/Category:Plateau-Rayleigh_instability#/media/File:Dripping_faucet_2.jpg

Instability phenomena are common and frequent, as for example a capillary jet breaking into droplets (instability of Rayleigh-Plateau, figure 2.1), where the surface tension plays an important role; the flow in a single continuous fluid with velocity shear or in two fluids with velocity difference at the interface (instability of Kelvin-Helmholtz, figure 2.2); density driven flows (instability of Rayleigh-Taylor) etc.

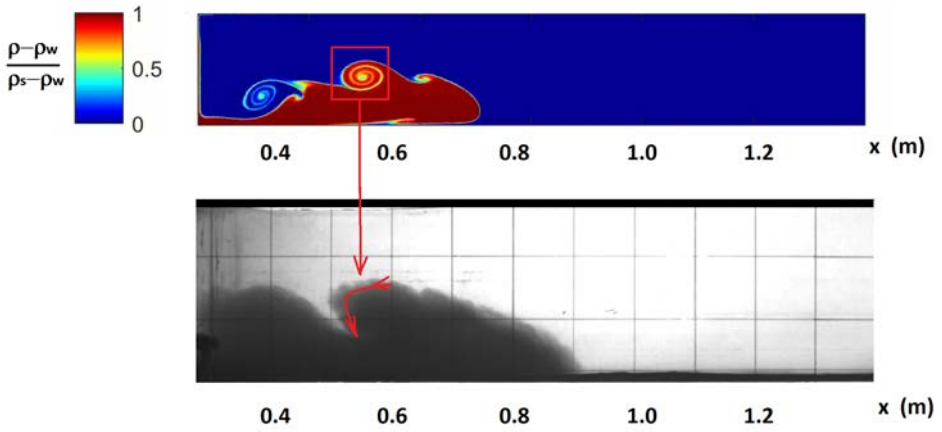

Figure 2.2: Computational (top) and experimental (bottom) Kelvin-Helmholtz instability in a gravity current.

The shear flow instability or Kelvin-Helmholtz instability is the most studied instability phenomenon. A shear flow is characterised by the fact that the velocity changes mostly with respect to the spatial coordinate perpendicular to the main direction of the flow. Examples of shear flows are the Poiseuille flow and the boundary layer. Kelvin-Helmholtz instability occurs when two fluid layers of the same fluid or of different fluids flow with different velocities u_1, u_2; i.e. they have a discontinuity in the velocity profile (figure 2.3).

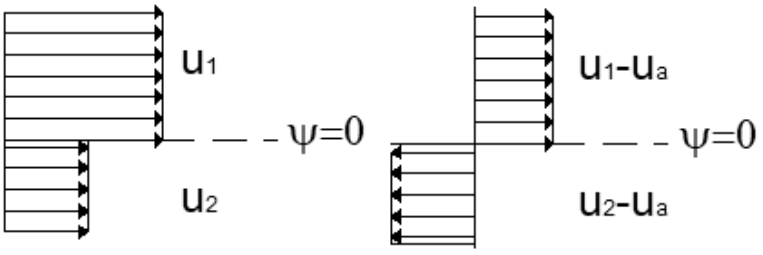

Figure 2.3: Discontinuity of the velocity profile at the separation surface of two fluid layers.

The justification for the Kelvin-Helmholtz instability, which had been given originally by some scientists in the XIX century based on hydrodynamic considerations and their knowledge, is the following. Let us consider the fluid as ideal (not viscous) and a frame of reference moving with the average velocity u_a of the two layers: $u_a = (u_1 + u_2)/2$. In this frame the fluid layers move away with opposite velocities $u_1 - u_a, u_2 - u_a$, being $u_1 > u_2$, while the separation surface $\psi = 0$ is a discontinuity surface. Let us suppose that a perturbation modifies the separation surface $\psi = 0$, so that the latter becomes a wavy surface (figure 2.4). Of course, the perturbation is extinguished moving from the separation surface $\psi = 0$, but in its neighbourhoods the streamlines are compressed in the convexities and thinned out in the concavities (figure 2.4). Assuming that the flow is steady, the pressure distribution along the separation surface $\psi = 0$, according to the Bernoulli's equation, increases in the concavities (the plus sign in figure 2.4), where the velocity decreases, and decreases in the convexities (the minus sign in figure 2.4), where the velocity increases, hence enhancing the waviness of the separation surface and making the flow unsteady.

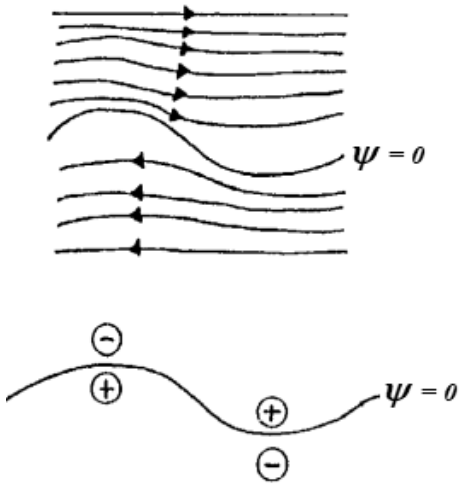

Figure 2.4: Wavy perturbation on the separation surface between two fluid layers traveling at different velocities.

In real, viscous fluids, the velocity discontinuity appears as a flex in the velocity profile, whose gradient is large in the so-called mixing layer around the discontinuity surface $\psi = 0$ and vanishing far from the latter (figure 2.5). Within the mixing layer, whose thickness increases along the direction of the flow, also the vorticity is very large and the dimensions of the vortices increase along the direction of the flow (figure 2.5).

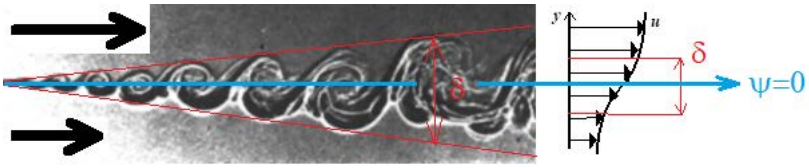

Figure 2.5: The mixing layer around the separation surface between two fluid layers traveling at different velocities.

Source: *On density effects and large structure in turbulent mixing layers*, Garry L. Brown and Anatol Roshko, Journal of Fluid Mechanics, Volume 64, Issue 04, July 1974, pp. 775-816, DOI: 10.1017/S002211207400190X.

The boundary layers with pressure gradients can have a "favourable" gradient, if the pressure decreases in the flow direction, or "adverse" gradient, if the pressure increases in the flow direction. The adverse pressure gradient provides an inflection point in the velocity profile.
The presence of flexes occurs in boundary layers too, when the flow separate from the surface. As shown in figure 2.6 when the velocity profile has a flex (profile C), the streamlines starts separating from the surface.

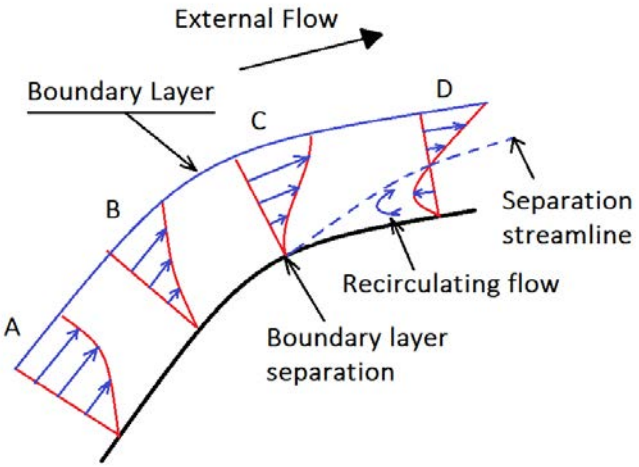

Figure 2.6: Separation of the boundary layer on a curved surface.

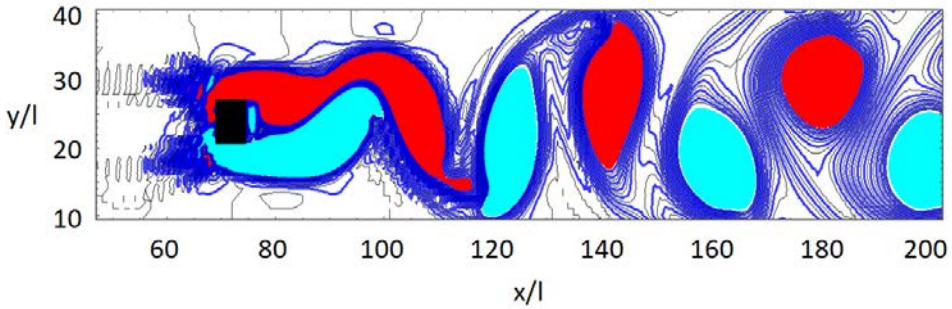

Figure 2.7: Von Karman Vortices resulting from Kelvin-Helmholtz instability in the wake downstream of a square cylinder, side l. Red: negative vorticity; light blue: positive vorticity. It is evident the generation and alternate detachment of the vortices downstream of the square cylinder.

Jets and wakes present flexes in velocity profiles too: that is why Kelvin-Helmholtz instabilities occur at the boundary of jets and wakes (figure 2.7).
As shown by Tollmien, in case of inviscid fluid flows, the presence of flexes in the velocity profile, as for jets or wakes, is a necessary condition, although not sufficient, for instability (Rayleigh's inflexion-point theorem).
Viscosity has generally a stabilizing effect, i.e. a damping effect on perturbations: this is the reason why, if the Reynolds number is small enough, the steady flow is stable, while it can become unstable for larger

Reynolds numbers even without inflection points in the velocity profile. In the latter case the viscosity determines the stability or not and it is therefore necessary to consider the viscosity in the stability analysis (Lin, 1955).

2.2. Linear Stability Analysis

In order to get qualitative information on the onset of instability and to give indications on the consequent flow, the theory of linear stability is the simplest and the most effective. The basis of this theory consists in considering the transition from one flow to another as the result of an amplification of small perturbations superimposed on an initial equilibrium state. The main idea of the linear theory is that the perturbations have to be small, with infinitesimal amplitude. Hence, the main limit of the linear theory is that it can only give a qualitative indication on the evolution of a considered flow subjected to small perturbations, while it cannot predict what happens when the perturbations become of finite amplitude. It is evident that a flow, which is stable, when subjected to infinitesimal perturbations, may become unstable when subjected to perturbations of finite amplitude. A flow that is unstable when subjected to infinitesimal perturbations is surely unstable when subjected to perturbations of finite amplitude.
The theory of linear stability consists of imposing on a given flow a small perturbation and in investigating if the amplitude of this perturbation amplifies or damps as time goes by.

2.2.1 Systems of first order ordinary differential equations

In order to explain how this theory works, let us consider a non-linear, first order, system of n ordinary differential equations:

$$\frac{dy_i}{dt} = f_i(y_1, y_2, \dots, y_n; \Pi), \quad i = 1,2, \dots, n \tag{2.2}$$

Where Π is a dimensionless parameter, e.g. the Reynolds number. The n functions f_i are assumed of class C^1. Let $y_1^0, y_2^0, \dots, y_n^0$ be a steady, equilibrium solution, obtained for $\Pi = \Pi^0$:

$$f_i(y_1^0, y_2^0, \dots, y_n^0; \Pi^0) = 0, \quad i = 1,2, \dots, n \tag{2.3}$$

Let us superimpose a perturbation $x_i = x_i(t)$ $(i = 1,2, \dots, n)$ on the steady equilibrium solution $y_1^0, y_2^0, \dots, y_n^0$. The evolution of this perturbation is governed by the equations:

$$\frac{dx_i}{dt} = f_i(x_1 + y_1^0, x_2 + y_2^0, \dots, x_n + y_n^0; \Pi^0), \quad i = 1,2, \dots, n \tag{2.4}$$

Let us expand in Taylor series the right hand side of equations 2.4, considering the steady equilibrium solution as initial point:

$$\frac{dx_i}{dt} = \left.\frac{\partial f_i}{\partial y_j}\right|_{y_i = y_i^0} x_j + \frac{1}{2}\left.\frac{\partial^2 f_i}{\partial y_j \partial y_k}\right|_{y_i = y_i^0} x_j x_k + \cdots \tag{2.5}$$

$$i = 1,2, \dots, n$$

Einstein's convention has been adopted in equations 2.5 for the sum of quantities having the same indices. Equations 2.5 are more conveniently put in matrix form:

$$\frac{d\mathbf{x}}{dt} = \mathbb{J}\mathbf{x} + \mathbf{r} \tag{2.6}$$

The term $\mathbb{J}\mathbf{x}$ is the first order term. The elements J_{ij} of the matrix $\mathbb{J}$ are defined as:

$$J_{ij} = \left.\frac{\partial f_i}{\partial y_j}\right|_{y_i = y_i^0} \tag{2.7}$$

while x_i $(i = 1,2, \dots, n)$ are the elements of the vector $\mathbf{x}$. Terms of order higher than the first are represented by the vector $\mathbf{r}$. Considering small perturbations, the ratio $|\mathbf{r}|/|\mathbf{x}|$ is negligible. Within this limit, the nonlinear differential system 2.5 can be linearized i.e. the high order terms represented by vector $\mathbf{r}$ can be neglected:

$$\frac{d\mathbf{x}}{dt} = \mathbb{J}\mathbf{x} \tag{2.8}$$

The solution of the linearized differential system 2.8 can be sought in the form:

$$\mathbf{x} = \mathbf{x}_0 e^{\sigma t} \tag{2.9}$$

$\mathbf{x}_0$ is the amplitude of the perturbation. Let us substitute the solution 2.9 in the linearized system 2.8. The following algebraic homogeneous system is obtained:

$$(\mathbb{J} - \sigma\mathbb{I})\mathbf{x}_0 = 0 \tag{2.10}$$

where $\mathbb{I}$ is the $n \times n$ identity matrix. The algebraic homogeneous system 2.10 has a nontrivial solution ($\mathbf{x}_0 \neq 0$) if the determinant of the matrix $\mathbb{J} - \sigma\mathbb{I}$ vanishes. This condition consists of an algebraic equation of n^{th} degree. The n roots of this equation are the eigenvalues of the matrix $\mathbb{J}$: they are complex numbers and can be expressed as:

$$\sigma = \sigma_R + I\sigma_I \tag{2.11}$$

Being I the imaginary unit σ_R, σ_I are the real and imaginary part of the eigenvalue σ respectively. The solution of the linearized differential system 2.8 is a linear combination of n exponential functions, each relative to the i^{th} eigenvalue. It follows that, if even only one eigenvalue has the real part positive ($\sigma_R > 0$), the perturbation increases indefinitely and the equilibrium solution is **unstable**. On the contrary, if the real part of every eigenvalue is

negative ($\sigma_R < 0$), the perturbation decreases indefinitely and the equilibrium solution is **stable**. Finally, if at least the real part of one eigenvalue vanishes, being the others negative, the perturbation does neither decrease, nor increase and the equilibrium solution is **marginally stable**.

2.2.2 The governing equations of a small disturbance to the flow

The continuity and the Navier-Stokes equations for an incompressible fluid subjected to gravity assume the following form:

$$\boldsymbol{\nabla}\cdot\mathbf{u}=0,\quad \frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}=-g\boldsymbol{\nabla}\zeta+\frac{\mu}{\rho}\boldsymbol{\nabla}^{\mathbf{2}}\mathbf{u} \tag{2.12}$$

Where ζ is the piezometric height ($\zeta=\frac{p}{\rho g}+x_3$).

Assume that $\mathbf{u},\zeta$ are solutions of equations (2.12).

Let us add perturbations $\mathbf{u}',\zeta'$ to the solutions $\mathbf{u},\zeta$, substitute the sum into equations (2.12), imposing that it is still a solution. We get:

$$\boldsymbol{\nabla}\cdot\mathbf{u}'=0,\quad \frac{\partial\mathbf{u}'}{\partial t}+\mathbf{u}'\cdot\boldsymbol{\nabla}\mathbf{u}+\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}'+\mathbf{u}'\cdot\boldsymbol{\nabla}\mathbf{u}'=-g\boldsymbol{\nabla}\zeta'+\frac{\mu}{\rho}\boldsymbol{\nabla}^{\mathbf{2}}\mathbf{u}' \tag{2.13}$$

The linear approximation of the Navier-Stokes equation, valid provided that the perturbations $\mathbf{u}',\zeta'$ are infinitesimal, consists in neglecting the convective term $\mathbf{u}'\cdot\boldsymbol{\nabla}\mathbf{u}'$ in the Navier-Stokes equation. Thus we get the linearized Navier-Stokes equation:

$$\frac{\partial\mathbf{u}'}{\partial t}+\mathbf{u}'\cdot\boldsymbol{\nabla}\mathbf{u}+\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}'=-g\boldsymbol{\nabla}\zeta'+\frac{\mu}{\rho}\boldsymbol{\nabla}^{\mathbf{2}}\mathbf{u}' \tag{2.14}$$

whose solutions can be considered approximations of those of equations (2.13), provided that the perturbations $\mathbf{u}',\zeta'$ are infinitesimal.

We can consider the perturbations $\mathbf{u}',\zeta'$ as the sum of spatial Fourier modes:

$$\mathbf{u}'=\sum_{\mathbf{k}}\mathbf{u}'_{\mathbf{k}}e^{I\mathbf{k}\cdot\mathbf{x}},\qquad \zeta'=\sum_{\mathbf{k}}\zeta'_{\mathbf{k}}e^{I\mathbf{k}\cdot\mathbf{x}} \tag{2.15}$$

Where $\mathbf{k}$ is the wavenumber vector and $\mathbf{x}$ is the position vector**.**

The time-dependent amplitudes of the Fourier mode, $\mathbf{u'_k}, \zeta'_\mathbf{k}$, are complex numbers. However, as $\mathbf{u}', \zeta'$ are real, the sums (2.15) are such that for each term they also include the complex conjugate, so that the final result is real.
The absence of the nonlinear term in the linearized Navier-Stokes equation (2.14) makes that each Fourier mode evolves over time independently of the other modes.
The time-dependent variables are the amplitudes $\mathbf{u'_k}, \zeta'_\mathbf{k}$ of the Fourier modes and their temporal evolution is governed by equations (2.14). The wave vector is considered as a parameter.
It is usual to assume an exponential temporal behaviour for the amplitudes $\mathbf{u'_k}, \zeta'_\mathbf{k}$ of the Fourier modes:

$$\mathbf{u'_k} = \mathbf{u'_{k0}} e^{\sigma_R t} e^{I(\mathbf{k}\cdot\mathbf{x}+\sigma_I t)}, \quad \zeta'_\mathbf{k} = \zeta'_\mathbf{k0} e^{\sigma_R t} e^{I(\mathbf{k}\cdot\mathbf{x}+\sigma_I t)} \tag{2.16}$$

Where $\mathbf{u'_{k0}}$ and $\zeta'_\mathbf{k0}$ are the values of $\mathbf{u'_k}$ and $\zeta'_\mathbf{k}$ for $t = 0$ and $\mathbf{x} = 0$.
According to (2.16), each Fourier mode oscillates with $\frac{\sigma_I}{2\pi}$ and amplifies or decreases with time depending on the sign of σ_R. The linearized governing equation 2.14 becomes the evolution equation of the amplitudes, which vary with the wave number, but do not depend on the spatial variables.
The linear stability analysis mentioned up to now consists, for each wavevector, in determining the parameter σ_R and therefore in deducing the temporal behaviour of the perturbations (stability analysis by means of the initial value of normal modes).
A simple consideration allows the results of the linear stability analysis to be verified experimentally and computationally.
We assume that the velocity vector is aligned with the x_1 axis and rewrite expressions (2.16) in the following way:

$$\begin{aligned} \mathbf{u'_k} &= \mathbf{u'_{k0}} e^{\sigma_R t} e^{I\left(k_1\left(x_1+\frac{\sigma_I}{k_1}t\right)+k_2x_2+k_3x_3\right)}, \\ \zeta'_\mathbf{k} &= \zeta'_\mathbf{k0} e^{\sigma_R t} e^{I\left(k_1\left(x_1+\frac{\sigma_I}{k_1}t\right)+k_2x_2+k_3x_3\right)} \end{aligned} \tag{2.17}$$

We introduce the new coordinate: $x_1^* = x_1 + \frac{\sigma_I}{k_1} t$. A constant value of x_1^* moves upstream along the old x_1 coordinate with velocity $c_I = \frac{\sigma_I}{k_1}$. Thus a perturbation, periodic in time with period $T = \frac{2\pi}{\sigma_I}$, moves upstream along the x_1 direction with velocity c_I and it becomes periodic in space along the x_1 coordinate with wavelength $c_I T$ and amplitude increasing or decreasing according to the sign of σ_R. In this way what happens in time is represented in space and the experimental verifications can be performed.
This approach is obtained also setting $\sigma_I = 0$ and considering the wavenumber complex (stability analysis by means of spatially growing modes).
However, both the stability analysis by means of the initial value of normal modes and spatially growing modes would lead to exhausting computational evaluations in order to consider all possible flows. Fortunately, the flows to be studied are often simple and allow us to assume useful simplifications in setting the problem. The linear stability analysis of plane incompressible flows, as we will see, can be performed through the Orr-Sommerfeld equation, which is further simplified to the Rayleigh's equation if the fluid can be considered as inviscid.

2.3. Orr-Sommerfeld equation

Let us extend the theory of linear stability to investigate the stability of a given incompressible, 2D, equilibrium flow occurring in the xy plane. The vorticity is a vector perpendicular to the xy plane, whose modulus ω is governed by the equation:

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{1}{Re}\nabla^2 \omega \tag{2.18}$$

Equation 2.18 is in dimensionless form. The theorem of Squire (1933) shows that every unstable three-dimensional disturbance corresponds to a more unstable two-dimensional disturbance. Thus even the disturbances can be considered two-dimensional. Let us express the vorticity modulus ω and the velocity components u, v, as the sum of the equilibrium solution and the perturbation:

$$u = U + u', v = V + v', \omega = \Omega + \omega' \tag{2.19}$$

Let us substitute the expressions 2.19 into the vorticity equation 2.18:

$$\frac{\partial \Omega}{\partial t} + \frac{\partial \omega'}{\partial t} + (U + u')\frac{\partial(\Omega + \omega')}{\partial x} + (V + v')\frac{\partial(\Omega + \omega')}{\partial y} = \frac{1}{Re}\nabla^2(\Omega + \omega') \tag{2.20}$$

The equilibrium solution satisfies the equation:

$$\frac{\partial \Omega}{\partial t} + U\frac{\partial \Omega}{\partial x} + V\frac{\partial \Omega}{\partial y} = \frac{1}{Re}\nabla^2\Omega \tag{2.21}$$

Thus the perturbation satisfies the equation:

$$\frac{\partial \omega'}{\partial t} + U\frac{\partial \omega'}{\partial x} + u'\frac{\partial(\Omega + \omega')}{\partial x} + V\frac{\partial \omega'}{\partial y} + v'\frac{\partial(\Omega + \omega')}{\partial y} = \frac{1}{Re}\nabla^2\omega' \tag{2.22}$$

Assuming that the perturbation quantities are small with respect to the equilibrium ones, equation 2.22 can be linearized, i.e. products of perturbations quantities can be neglected:

$$\frac{\partial \omega'}{\partial t} + U\frac{\partial \omega'}{\partial x} + u'\frac{\partial \Omega}{\partial x} + V\frac{\partial \omega'}{\partial y} + v'\frac{\partial \Omega}{\partial y} = \frac{1}{Re}\nabla^2\omega' \tag{2.23}$$

Let us focus on a 2D flow such that:

$$U = U(y), \qquad V = 0, \qquad \Omega = -\frac{dU}{dy} \tag{2.24}$$

The perturbation could be 3D but the Squire's theorem (1933) states that the critical Reynolds number occurs for two-dimensional perturbations. So we can restrict the analysis to these ones and to express the perturbation velocity components by means of a stream function:

$$\psi = \psi(x, y, t): \tag{2.25}$$

and the only not vanishing component of the perturbation vorticity can be expressed in terms of the perturbation velocity components as:

$$u' = \frac{\partial \psi}{\partial y}, \; v' = -\frac{\partial \psi}{\partial x} \;\Rightarrow\; \omega = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) \tag{2.26}$$

Then equation 2.21 becomes:

$$\frac{\partial \nabla^2 \psi}{\partial t} + U\frac{\partial \nabla^2 \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{d^2 U}{dy^2} = \frac{1}{Re}\nabla^4 \psi \tag{2.27}$$

Equation 2.21 is the linearized vorticity equation, expressed in terms of the known equilibrium velocity profile $U = U(y)$ and the unknown stream function. The latter can be expressed as:

$$\psi = \Phi(y) e^{I\, k_x (x - ct)} \tag{2.28}$$

The structure of the function 2.28 is a traveling wave along direction x, with celerity c and wavenumber k_x, corresponding to the wave length λ_x ($\lambda_x = 2\pi/k_x$). The amplitude of the wave is the unknown function $\Phi = \Phi(y)$, which has to be determined by solving equation 2.27. Substitution of the expression 2.28 into equation 2.27 gives:

$$\left((c - U)\left(\frac{d^2}{dy^2} - k_x^2\right) + \frac{d^2 U}{dy^2}\right)\Phi = \frac{I}{k_x Re}\left(\frac{d^2}{dy^2} - k_x^2\right)^2 \Phi \tag{2.29}$$

Equation 2.29 is the **Orr-Sommerfeld equation**. It is a fourth order, linear, ordinary differential equation with non-constant coefficients. Non trivial solution Φ (i.e. $\Phi \neq 0$) are defined eigenfunctions of the Orr-Sommerfeld equation and exist for well defined values of c, defined as eigenvalues of the Orr-Sommerfeld equation, given the wavelength k_x and the Reynolds number Re.

Boundary conditions have to be imposed. Let us assume that at $y = 0$ and at $y = H$ there are two solid walls: it follows that at $y = 0$ and at $y = H, u', v'$ vanish. In terms of the function Φ:

$$u'|_{y=0,H} = \left.\frac{\partial \psi}{\partial y}\right|_{y=0,H} = \left.\frac{d\Phi}{dy}\right|_{y=0,H} e^{I\,k_x(x-ct)} = 0 \;\Rightarrow\; \left.\frac{d\Phi}{dy}\right|_{y=0,H} = 0$$

$$v'|_{y=0,H} = -\left.\frac{\partial \psi}{\partial x}\right|_{y=0,H} = -Ik_x\Phi|_{y=0,H} e^{I\,k_x(x-ct)} = 0 \;\Rightarrow\; \Phi|_{y=0,H} = 0 \tag{2.30}$$

Given the wavelength and the Reynolds number, the eigenvalues of the Orr-Sommerfeld equation characterize the stability of the equilibrium solution. They are complex numbers. According to the structure 2.28 of the perturbation, the stability properties are determined by the imaginary part of the eigenvalues. If, given the wavelength and the Reynolds number, all of the eigenvalues have negative imaginary part, the perturbation is damped and the equilibrium flow is stable, while if even one eigenvalue has positive imaginary part, the perturbation is amplified and the equilibrium flow is unstable. If we set $Re = \infty$ in the Orr-Sommerfeld equation, the right hand side vanishes and an equation valid for inviscid fluids, known as Rayleigh's equation, is obtained. This reduced equation has been studied for a long time, obtaining theorems as the already cited Rayleigh's inflexion-point theorem or the Fjortoft's theorem stating that the magnitude of vorticity of the basic flow must have a maximum within the region of flow, not at the boundary.

2.4. Marginal stability curves

It is usual to represent the stability characteristics of the equilibrium flow by means of stability curves in the k_x, Re plane. These curves are also known as marginal stability curves and represent the condition when at least one eigenvalue has vanishing imaginary part and the imaginary parts of the other eigenvalues are negative. The marginally stability curves separate the instability from the stability zones. Typical shapes of the marginal stability curves are shown in figure 2.8. The dashed curve is relative to a flow with a flex in the velocity profile, while the solid curve is relative to a velocity profile without flex. In fact, the stability theory predicts stability for an ideal fluid flow ($Re \to \infty$) without flex in the velocity profile, while it predicts instability for an ideal fluid flow ($Re \to \infty$) with flex in the velocity profile. In both cases (solid and dashed curve), there is a critical Reynolds number Re_{cr}, i.e. a minimum value of the Reynolds number at which the instability occurs. A critical vanishing wave number ($k_{cr} = 0$) can occur only if the fluid domain – and then the largest wavelength – is unlimited. The value of the critical Reynolds number depends on the considered flow: for the Poiseuille flow $Re_{cr} = 5770$, while for the flow within a pipe with circular section $Re_{cr} \approx 2000 \div 2500$.

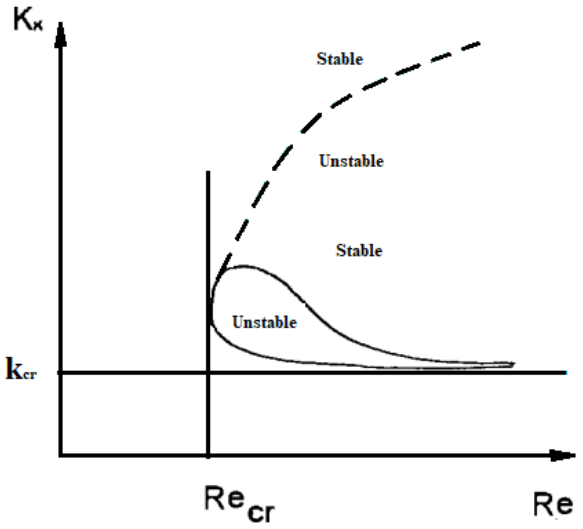

Figure 2.8: Typical shapes of the marginal stability curves in the k_x, Re plane.

Two types of instability mechanisms can be identified: inviscid instabilities with a flex in the velocity profile, where viscosity is not important in the onset of turbulence, and viscous instabilities, where viscosity plays a crucial role in the transition to turbulence. In the first category, the presence of inflection denotes the presence of maximum vorticity within the flow. This situation is unstable: vortices arise continuously, increase their intensity and generate other vortices, whose characteristic length is smaller than that of the generating ones. As a consequence, a generation of smaller and smaller vortices occurs. Viscosity plays no role in this process; it becomes effective only in the final stage.

In the second category, the vorticity does not achieve a maximum within the flow but, as in the Poiseuille or the Couette flow, it does on the boundary. Vorticity here does not play a prominent role in the initial instabilities, while the viscosity does. This is contrary to expectations as we think to viscous effects as of dissipative nature and therefore inherently stabilizing. The reason is that fluctuations in viscous stresses, caused e.g. by a wave disturbance, can present phase delays that overlap the wave disturbance, amplifying it by means of a resonance-like phenomenon.

An interesting example of viscous instability is that which occurs in the laminar boundary layer as the Reynolds number increases. The case of the boundary layer on a flat plate is particularly interesting, because the Reynolds number increases locally with the spatial coordinate defined along the flat plate. It follows that the boundary layer becomes unstable only after a given position on the flat plate, where the value of the local Reynolds number is greater than the critical value. At low Reynolds numbers, the viscous forces are so strong that they dampen any disturbances in the flow and thus the boundary layer remains laminar. As the Reynolds number increases, inertial effects grow and become dominant causing boundary layer flow instability. Above the Reynolds critical number, arise the two-dimensional perturbations of the boundary layer, known as Tollmien-Schlichting waves, traveling in the main direction of the flow. These waves are slowly amplified as they travel downstream until they become so large that non-linearities take over. In flows starting with almost no natural velocity fluctuations, turbulence is initiated by these waves. The Tollmien-Schlichting waves are defined as the most unstable eigen-modes of the Orr-Sommerfeld equations.
Schubauer e Skramstad in a famous experiment performed in 1947 in a low velocity wind tunnel have proved this fact (figure 2.9).

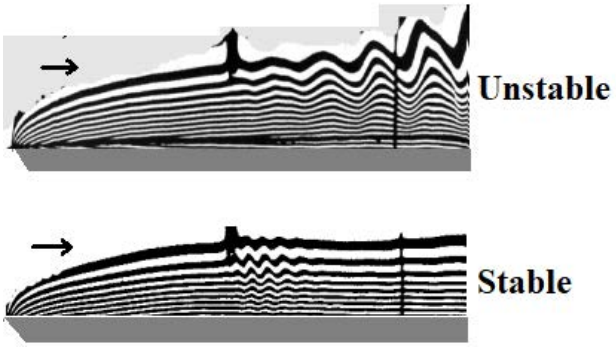

Figure 2.9: Stable and unstable boundary layer on a flat plate.
In both cases, the so-called Tollmien-Schlichting waves are visible.

Source: Physical Fluid Dynamics, Tritton, D. J. Springer, 1977,
ISBN 10: 0442301324ISBN 13: 9780442301323.

The resolution of the hydrodynamic stability problem allows to determine the onset/triggering of the transition mechanism to turbulent flow but does not answer the crucial question of how random turbulent fluctuations develop from oscillatory laminar flows

Among the different scenarios of transition to turbulence, the one proposed by Landau (1944) and Hopf (1948), the most accredited until the mid-1970s (until the discover of the *strange attractors*), states that as the Reynolds number increases, after the first unstable mode, an increasing number of unstable Fourier modes develops. Among the unstable modes there are couples, whose ratios between the periods of oscillation are irrational and whose superposition is not periodic but quasi-periodic. This means that the sequence of values never repeats exactly.

From the mathematical point of view a bifurcation occurs when, for a given value of a relevant parameter, several different solutions develop, e.g. passing from a stationary to one or more periodic solutions. The bifurcation theory arose with J.H. Poincaré (1854-1912) and then became the subject of intense research by V. Arnold, J. Marsden and other researchers. The representation of the characteristic properties of the solutions of the motion equations in terms of the governing parameters, as the Reynolds number, constitutes the bifurcation diagram. As the governing parameters vary, bifurcations follow one another in a process known as the Feigenbaum cascade, after the american physicist M. J. Feigenbaum. According to Landau, the process of transition to the turbulent regime is interpreted as a sequence of successive bifurcations of the solution of the motion equations. Some of the solutions downstream of a bifurcation may be unstable, from which it can be inferred that flow instability can be interpreted as a bifurcation of the analytical solution. Representing the flow as a superposition of travelling waves (equation 2.28), the number of bifurcations rapidly increases, thus determining the chaotic behaviour, whose representation requires a suitably high number of waves. This scenario has been questioned by the discovery of dynamic systems, which exhibit chaotic behaviour even with a very low number of waves (strange attractors theory).

CHAPTER 3 CHARACTERISTICS OF TURBULENT FLOWS

3.1. Laminar and turbulent flow

Let us imagine a flow with steady boundary conditions, whose viscosity can be changed. It is known that, if the viscosity is high enough and consequently the Reynolds number is low enough, the flow is steady, but as soon as the viscosity assumes low values and consequently the Reynolds number becomes high enough, even if the boundary conditions are steady, the flow becomes unstable and unsteady. As the Reynolds number increases, the flow becomes extremely irregular and chaotic. The velocity components oscillate without any regularity and the pathlines of two fluid particles, which were very close at the beginning of motion, are extremely different and twisted. This process is, in a nutshell, the laminar-turbulent transition. In figure 3.1 a turbulent flow in a diverging channel is made visible by means of small hydrogen bubbles, injected into the flow, which follow the fluid particle.

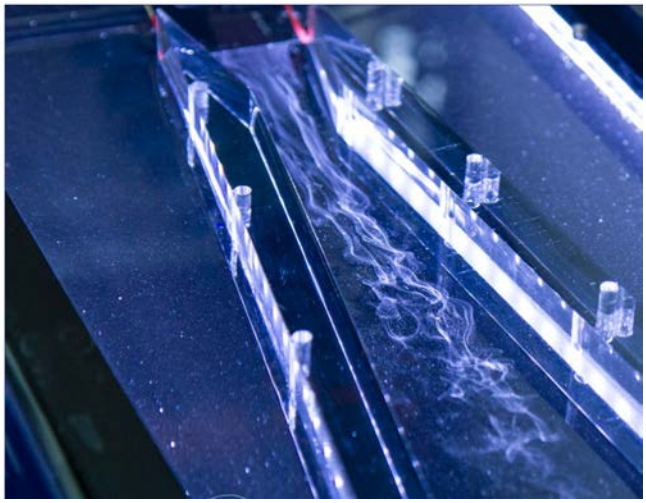

Figure 3.1: Typical disordered and chaotic turbulent water flow made visible by means of small hydrogen bubbles.

Source: https://armfield.co.uk/wp-content/uploads/2020/03/C16_Datasheet_v2e_ web.pdf.

The word *turbulent* evokes something messy, irregular and chaotic. On the contrary, the word *laminar* means well ordered, regular and predictable. These are not rigorous definitions of laminar and turbulent flows. A more rigorous approach refers to the role of the initial conditions. Indeed, if the viscosity is sufficiently high, i.e. if the Reynolds number is sufficiently small, starting from any initial conditions, the flow, after a transient, will tend to a state depending only on the boundary conditions and independent of the initial conditions. This flow is laminar. For smaller viscosity values, i.e. for greater Reynolds numbers, the transient lengthens without limit: this means that the flow depends not only on the boundary conditions but also on the initial conditions. For even smaller viscosity values and correspondingly larger Reynolds numbers, the flow will depend strongly on the initial conditions. This flow is turbulent. In actual flows it is unavoidable that small variations of the initial conditions occur in different experiments with the same Reynolds number and boundary conditions. Then if the influence of the initial conditions is strong, even in the same experiment two originally very close fluid particles with different initial conditions develop very different trajectories. In conclusion, the difference between a laminar or turbulent flow can be seen in the vanishing or strong dependence of the flow on the initial conditions.

From an analytical point of view, the Navier-Stokes equation is a valid tool in investigating turbulent flows, as it has been proven that, under appropriate hypotheses generally verified in most cases, their solution is unique. Therefore a laminar flow corresponds to the stable asymptotic solution of the equation, not dependent on initial conditions; while a turbulent flow corresponds to an infinite transient of the solution, always dependent on the initial conditions[2]. The problem for turbulent flows is the exact knowledge of the initial conditions.

From a phenomenological point of view, any scenario that attempts to interpret the turbulent phenomenon must consider experimental evidences.

[2] In this regard, see Joseph B.D. *Stability of Fluid Motions*, Spriger Verlag, 1975.

The consolidated experimental data indicate that the turbulent flow is three-dimensional and rotational with vorticity also irregularly and disorderedly distributed in the three spatial directions. The mixing and diffusion processes are highly developed. Fourier analysis of kinematic and dynamic variables of turbulent flows shows that frequency or wavenumber spectra tend to depend with continuity on frequency or wavenumber, indicating that there is no temporal or spatial periodicity. The autocorrelation coefficients of the turbulent quantities tend to zero rapidly with the increase of the phase shift, as happens with the random variables, not allowing reliable predictions on the future trend of the variables from the knowledge of the past trend.

3.2. Developed turbulence and predictability

Let us consider the measurement of any velocity component, performed at a given point during a sufficiently large time interval (figure 3.2).

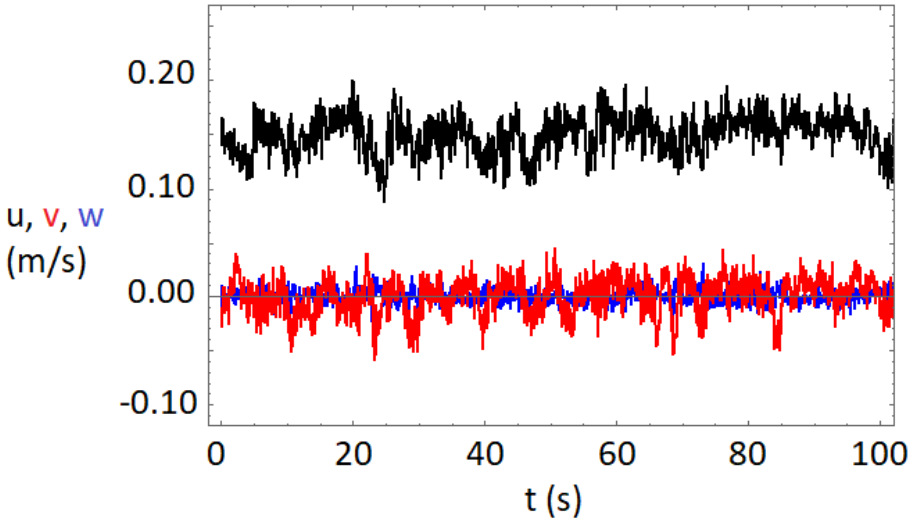

Figure 3.2: Measurement of u,v,w velocity components in a turbulent open channel flow.

The frequency spectrum of the time history of the velocity component shown in figure 3.2 can be obtained by means of the Fourier analysis. If the spectrum consists of a continuous distribution of frequencies, the flow can be considered to be turbulent, while if isolated peaks are present, the flow is periodic (or quasi-periodic)[3] and predictable.
The laminar or turbulent character of the flow is highlighted by the auto-correlation coefficient:

$$C_{uu}(\theta) = \frac{\int_{-\infty}^{+\infty} u(t')u(t'+\theta)dt'}{\int_{-\infty}^{+\infty} u(t')^2 dt'} \tag{3.1}$$

being $u(t)$ a time dependent velocity component at a given point and θ the delay time. The auto correlation coefficient gives an indication of the predictability of the flow velocity. If the velocity component is periodic or quasi periodic, its frequency spectrum consists of isolated rows. A quasi-periodic function is characterised by a frequency spectrum with nonzero rows, corresponding to frequencies, whose ratio is an irrational number.
The autocorrelation coefficient is periodic or quasi-periodic with respect to the delay time, achieving maxima and minima at periodic intervals of time (figure 3.3). In this case the flow velocity is predictable and so the flow as a whole[4]. If the velocity component has a frequency spectrum with a continuous distribution of frequencies, the autocorrelation coefficient vanishes as the delay time increases (figure 3.4) and the flow velocity is unpredictable and so the flow as a whole. It follows that a turbulent flow, being unpredictable, can be considered to be the sum of an infinite number of harmonic functions, with a continuous distribution of frequencies.

[3] If the ratio of the frequencies of the peaks is not rational.
[4] For this reason, the scenario of transition to turbulence hypothesized by Landau and Hopf is now considered not satisfactory.

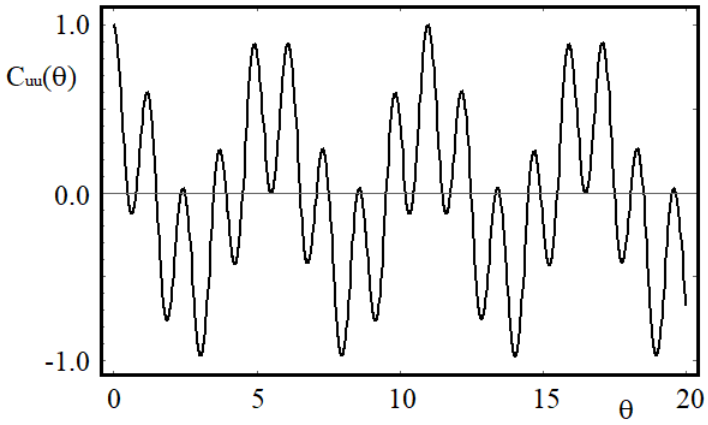

Figure 3.3: Behaviour of the autocorrelation function with respect to the delay time for a quasi-periodic signal.

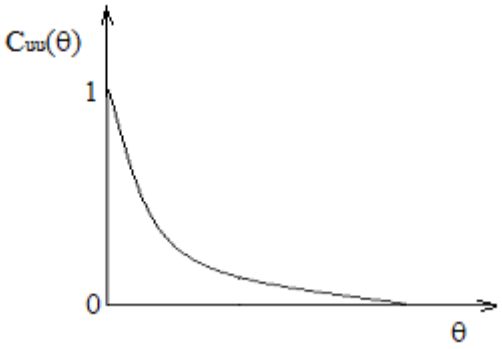

Figure 3.4: Behaviour of the autocorrelation function with respect to the delay time for a random signal.

3.3. The Reynolds average approach to turbulence

Let us consider any quantity per unit of mass expressed by the variable f. The balance equation has the general form:

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} f = \boldsymbol{\nabla} \cdot (\boldsymbol{\Gamma}\, \boldsymbol{\nabla} f) + \mathrm{S} \tag{3.2}$$

Being S a source or sink term and **Γ** the diffusivity matrix. The magnitude of the elements of the latter gives an indication of the tendency of the considered quantity to diffuse. The greater the magnitude, the greater the tendency to diffuse.
The velocity field **u** affects the behaviour of the considered flow quantity f: if the former fluctuates randomly, the latter behaves the same. The quantity f may in turn affect the velocity field: in this case it is defined as active quantity, while if it does not affect the velocity field it is defined passive.
The balance equation 3.2 is impossible to solve for f if the velocity field **u** is turbulent: it is more convenient to follow a statistical approach based on the hypothesis to have n subsequent executions or realizations of the same turbulent flow, obtained being equal the boundary conditions. Initial conditions can change from a realization to the other, being affected by small perturbations. The i^{th} realization of the flow gives the quantity f_i=$f_i(\mathbf{x}, t)$, at the point **x** and the time t. Let us define the ensemble average[5] as:

$$\langle f(\mathbf{x}, t)\rangle = \frac{1}{n}\sum_{i=1}^{n} f_i(\mathbf{x}, t) \tag{3.3}$$

Of course, the number of realizations n has to be large enough, so that the results of the ensemble operations do not depend on n.
The ensemble average is a linear operation, i.e. satisfies the following properties, given a constant c and another flow quantity $h(\mathbf{x}, t)$:

$$\langle cf(\mathbf{x}, t)\rangle = c\langle f(\mathbf{x}, t)\rangle,\ \langle f(\mathbf{x}, t) + h(\mathbf{x}, t)\rangle = \langle f(\mathbf{x}, t)\rangle + \langle h(\mathbf{x}, t)\rangle \tag{3.4}$$

The residual $f_i'(\mathbf{x}, t)$ is defined as the difference of the single realization with the ensemble average:

$$f_i'(\mathbf{x}, t) = f_i(\mathbf{x}, t) - \langle f(\mathbf{x}, t)\rangle \tag{3.5}$$

[5] Actually Osborne Reynolds considered the temporal mean in his approach but today is preferable to use the ensemble average.

Then the ensemble average of the residual vanishes identically:

$$\langle f'(\mathbf{x},t)\rangle = \langle f(\mathbf{x},t)\rangle - \langle\langle f(\mathbf{x},t)\rangle\rangle = \langle f(\mathbf{x},t)\rangle - \langle f(\mathbf{x},t)\rangle = 0 \tag{3.6}$$

Indeed the ensemble average of the ensemble average gives simply the ensemble average.

The ensemble average of the product of two quantities f, h, expressing them as: $f = \langle f\rangle + f', h = \langle h\rangle + h'$, gives:

$$\langle fh\rangle = \langle(\langle f\rangle + f')(\langle h\rangle + h')\rangle = \langle\langle f\rangle\langle h\rangle\rangle + \langle\langle f\rangle h'\rangle + \langle f'\langle h\rangle\rangle + \langle f'h'\rangle =$$

$$\langle f\rangle\langle h\rangle + \langle f\rangle\langle h'\rangle + \langle f'\rangle\langle h\rangle + \langle f'h'\rangle = \langle f\rangle\langle h\rangle + \langle f'h'\rangle \tag{3.7}$$

The ensemble average of the product of two quantities is expressed as the sum of the product of the ensemble average of the two quantities and the ensemble average of the product of the residuals. The latter is generally not vanishing.

Finally, differentiation does neither affect nor is affected by the ensemble average operation:

$$\langle\frac{\partial f}{\partial t}\rangle = \frac{\partial\langle f\rangle}{\partial t}, \quad \langle\boldsymbol{\nabla} f\rangle = \boldsymbol{\nabla}\langle f\rangle \tag{3.8}$$

In previous considerations, f, h have been implicitly considered to be scalars, but the same results would have been obtained with vector quantities. Equations 3.3-3.8 express the Reynolds conditions.

Let us apply the ensemble average operator to the mass conservation equation for an incompressible flow:

$$\langle\boldsymbol{\nabla}\cdot\mathbf{u}\rangle = \langle 0\rangle \quad \Rightarrow \quad \boldsymbol{\nabla}\cdot\langle\mathbf{u}\rangle = 0 \tag{3.9}$$

The ensemble average velocity field is solenoidal, so it follows that the residual velocity field is also solenoidal:

$$\langle \boldsymbol{\nabla} \cdot \mathbf{u} \rangle = \langle \boldsymbol{\nabla} \cdot (\langle \mathbf{u} \rangle + \mathbf{u}') \rangle = \boldsymbol{\nabla} \cdot \langle \mathbf{u} \rangle + \boldsymbol{\nabla} \cdot \langle \mathbf{u}' \rangle = 0 \tag{3.10}$$

Accounting for equation 3.9, follows:

$$\boldsymbol{\nabla} \cdot \langle \mathbf{u}' \rangle = 0 \tag{3.11}$$

3.3.1 The turbulent diffusivity

Let us apply the ensemble average operator to the general form of the balance equation 3.2:

$$\langle \frac{\partial f}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} f \rangle = \langle \boldsymbol{\nabla} \cdot (\boldsymbol{\Gamma}\, \boldsymbol{\nabla} f) + \mathrm{S} \rangle \tag{3.12}$$

Assuming that the source or sink term S and the diffusivity matrix $\boldsymbol{\Gamma}$ have the same values for each realization and accounting for properties 3.8, the ensemble average balance equation takes the form:

$$\frac{\partial \langle f \rangle}{\partial t} + \langle \mathbf{u} \cdot \boldsymbol{\nabla} f \rangle = \langle \boldsymbol{\nabla} \cdot (\boldsymbol{\Gamma}\, \boldsymbol{\nabla} f) \rangle + \langle \mathrm{S} \rangle \tag{3.13}$$

Thanks to the property 3.7, the ensemble average of the convective term at the left hand side of equation 3.13 can be expressed as:

$$\langle \mathbf{u} \cdot \boldsymbol{\nabla} f \rangle = \langle \mathbf{u} \rangle \cdot \langle \boldsymbol{\nabla} f \rangle + \langle \mathbf{u}' \cdot \boldsymbol{\nabla} f' \rangle \tag{3.14}$$

Moreover, thanks to the property 3.7, the ensemble average of the convective term at the left hand side of equation 3.13 can be expressed as:

$$\langle \mathbf{u} \cdot \boldsymbol{\nabla} f \rangle = \boldsymbol{\nabla} \cdot \langle \mathbf{u} \rangle \langle f \rangle + \boldsymbol{\nabla} \cdot \langle \mathbf{u}' f' \rangle \tag{3.15}$$

equation 3.13 assumes consequently the form:

$$\frac{\partial \langle f \rangle}{\partial t} + \boldsymbol{\nabla} \cdot \langle \mathbf{u} \rangle \langle f \rangle = \boldsymbol{\nabla} \cdot (\boldsymbol{\Gamma}\, \boldsymbol{\nabla} \langle f \rangle) - \boldsymbol{\nabla} \cdot \langle \mathbf{u}' f' \rangle + \langle \mathrm{S} \rangle \tag{3.16}$$

With respect to the original form of the balance equation 3.2, the ensemble average of the balance equation 3.16 has the additional term $-\boldsymbol{\nabla} \cdot \langle \mathbf{u}' f' \rangle$ at Right Hand Side. This term is known as the turbulent transport term. The latter is unknown since it is expressed in terms of the residuals of the velocity field and the flow quantity f.
The ensemble average of the balance equation 3.16 can then be solved in order to determine the ensemble average of the flow quantity f, once the ensemble average velocity field $\langle \mathbf{u} \rangle$ is known, only if the turbulent transport term can be expressed in terms of known quantities: i.e. if a suitable closure hypothesis is introduced.
A very formal closure hypothesis consists of introducing the turbulent diffusivity matrix $\boldsymbol{\Gamma}'$ and expressing the ensemble average of the product $\langle \mathbf{u}' f' \rangle$ in terms of the gradient of the ensemble average of the flow quantity f:

$$\langle \mathbf{u}' f' \rangle = -\boldsymbol{\Gamma}' \boldsymbol{\nabla} \langle f \rangle \tag{3.17}$$

As a consequence, the ensemble average equation can be put in the form:

$$\frac{\partial \langle f \rangle}{\partial t} + \boldsymbol{\nabla} \cdot \langle \mathbf{u} \rangle \langle f \rangle = \boldsymbol{\nabla} \cdot (\boldsymbol{\Gamma}\, \boldsymbol{\nabla} \langle f \rangle) + \boldsymbol{\nabla} \cdot (\boldsymbol{\Gamma}'\, \boldsymbol{\nabla} \langle f \rangle) + \langle \mathrm{S} \rangle \tag{3.16}$$

In other words the turbulent transport term is represented by an additional diffusive term, whose diffusivity matrix $\boldsymbol{\Gamma}'$ is not known. The turbulent diffusivity has been introduced by Boussinesq: it depends on the flow field and is generally greater than the molecular diffusivity.
The Boussinesq hypothesis represents the effect of the turbulent transport, due to the fluctuating convective motion, as a strong tendency to dilute and diffuse the flow quantities.

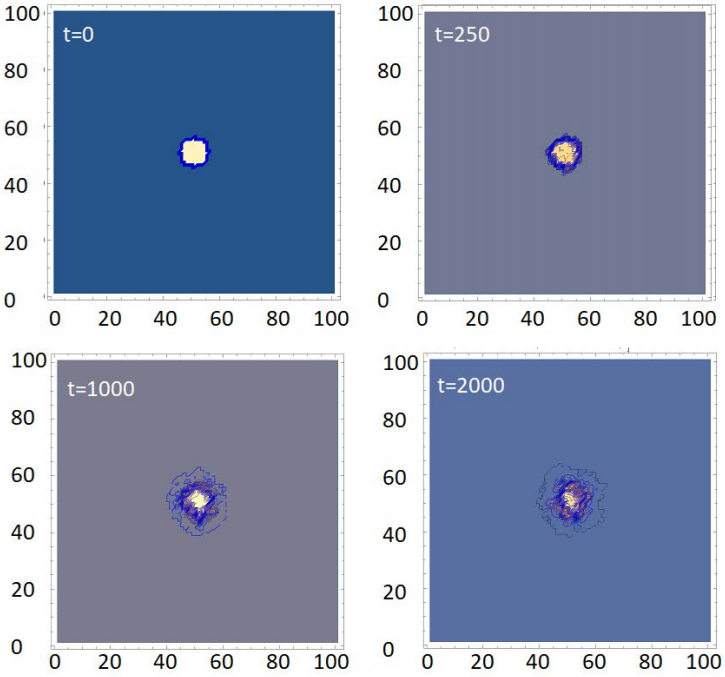

Figure 3.5: Effect of the turbulent transport on a passive scalar initially confined in a circular region.

In figure 3.5 the effect of the turbulent transport is shown for the concentration of a passive scalar, immersed in a homogeneous and isothropic turbulent velocity field. Time and spatial scales are dimensionless. The region occupied by the passive scalar at time t=0 is circular. As time goes by, it takes on an increasingly complex and irregular shape due to the action of the turbulent velocity field.

3.4. Time average and expected value. Scales of the turbulent motion.

The ensemble average operation is conceptually important, but practically impossible to realize, as it needs a very large number of repeated experiments. From a single experiment however, it is possible to obtain time

and spatial averages. Let us define the time average of a time dependent flow quantity f as:

$$\bar{f} = \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} f(t')dt' \tag{3.19}$$

It is generally assumed that, if T is large enough and the boundary conditions are steady, the time average turbulent variable will be steady too, i.e. it will neither depend on t nor on T. In other words the time average operation filters out the turbulent fluctuations from a turbulent quantity.
The time average can be used instead of the ensemble average, with an evident advantage, in order to obtain the average balance equation, if the value of the integral:

$$\Lambda_t = \int_0^{\infty} \mathrm{C}_{uu}(\theta)d\theta \tag{3.20}$$

is finite. Λ_t is dimensionally homogeneous to an interval of time. The condition 3.20 is satisfied if the autocorrelation coefficient tends to zero with the increase of the delay time θ: i.e. if the relative flow quantity loses correlation with itself after a while. In this case, if one samples the flow quantity for an interval of time T, so that $T \gg \Lambda_t$, it is as if one got many different realizations of the same experiment, because the internal correlation of the flow quantity is completely lost after a time interval Λ_t, so the time average can be considered equivalent to the ensemble average. The time interval Λ_t is the turbulent integral time-scale. Let us consider the time behaviour of the autocorrelation coefficient relative to an unpredictable, turbulent flow quantity, shown in figure 3.6.
The meaning of the turbulent integral time-scale is the basis of the rectangle, the area of which is equivalent to that subtended by the dashed line.

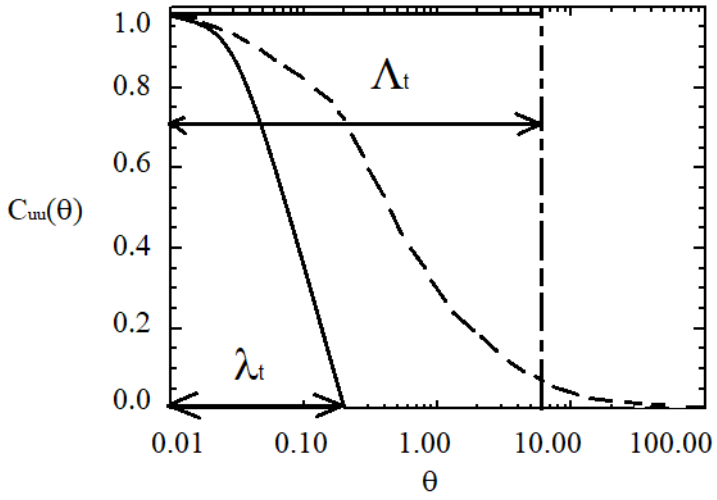

Figure 3.6: The meaning of the integral turbulent time-scale Λ_t and of the turbulent time-microscale λ_t.

Let us consider the osculating parabola to the function of the autocorrelation coefficient $C_{uu}(\theta)$ at the point $\theta = 0, \boldsymbol{C}_{uu} = 1$. According to the parabola, the autocorrelation coefficient vanishes for $\theta = \lambda_t$, given by the algebraic expression:

$$\lambda_t = \sqrt{-\frac{2}{\frac{d^2 C_{uu}}{d\theta^2}\Big|_{\theta=0}}}$$

The time interval λ_t is defined as the turbulent time-microscale and is representative for the internal coherence of the flow quantity, i.e. it represents the time interval during which the flow quantity shows correlation with itself. The predictability of any given turbulent flow quantity, whose turbulent time-microscale is λ_t, disappears for $t > \lambda_t$. Spatial turbulence scales can be defined in a similar way.
An important quantity is the time average of the square of the residuals of the flow quantity:

$$\overline{f'^2} = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} \left(f(t') - \bar{f}\right)^2 dt' \tag{3.23}$$

it gives the order of magnitude of the variation of the turbulent quantity.
An alternative and interesting definition of average can be given in terms of probability density function (PDF) $\mathfrak{p} = \mathfrak{p}(f)$ relative to the turbulent flow quantity. The meaning of the PDF is that $d\mathfrak{p} = \mathfrak{p}(f)df$ gives the probability, i.e. the degree of feasibility expressed in percentage, that the turbulent flow quantity f assumes a value belonging to the interval $f, f + df$. The fundamental property of the PDF is:

$$\int_{-\infty}^{+\infty} \mathfrak{p}(f)df = 1 \tag{3.24}$$

i.e. the probability that the turbulent flow quantity assumes values belonging to the interval $-\infty < f < \infty$ is 100%.
The average or expected value f_μ is defined as the first statistical moment of the PDF, with respect to f:

$$f_\mu = \int_{-\infty}^{+\infty} f\mathfrak{p}(f)df \tag{3.25}$$

while the second statistical moment of the PDF with respect to $f - f_\mu$ gives σ^2, the means square of the residual:

$$\sigma^2 = \int_{-\infty}^{+\infty} \left(f - f_\mu\right)^2 \mathfrak{p}(f)df \tag{3.26}$$

The shape of the PDF is characteristic for the considered phenomenon.

CHAPTER 4 THE REYNOLDS AVERAGE NAVIER-STOKES

4.1. The Reynolds equation (RANSE) for incompressible flows

In the previous chapter we have shown how applying the ensemble average operation to the mass conservation equation for an incompressible flow, the mass conservation equation for the ensemble average flow 3.9 is obtained. Moreover, the ensemble average operation has been applied to the general form of the balance equation 3.2 and the equation 3.16 has been obtained. Let us consider the momentum equation of a viscous Newtonian, incompressible fluid, i.e. the Navier-Stokes equation, and apply the ensemble average operation term by term:

$$\langle \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} \rangle = - \langle \frac{1}{\rho} \boldsymbol{\nabla} p \rangle + \langle \mathbf{f}_b \rangle + \langle \frac{\mu}{\rho} \boldsymbol{\nabla}^2 \mathbf{u} \rangle \tag{4.1}$$

Let us observe that, due to incompressibility and the conditions 3.8, the first term at right hand side gives:

$$\langle \frac{1}{\rho} \boldsymbol{\nabla} p \rangle = \frac{1}{\rho} \langle \boldsymbol{\nabla} p \rangle = \frac{1}{\rho} \boldsymbol{\nabla} \langle p \rangle \tag{4.2}$$

The second term at right hand side, assuming that the body force per unit mass $\mathbf{f}_b$ assumes the same value for each realization (as e.g. the gravity force), gives:

$$\langle \mathbf{f}_b \rangle = \mathbf{f}_b \tag{4.3}$$

The third term at right hand side, thanks to the interchangeability of the ensemble average and differential operations, gives:

$$\langle \frac{\mu}{\rho} \boldsymbol{\nabla}^2 \mathbf{u} \rangle = \frac{\mu}{\rho} \boldsymbol{\nabla}^2 \langle \mathbf{u} \rangle \tag{4.4}$$

For the same reason, the first term at left hand side gives:

$$\langle \frac{\partial \mathbf{u}}{\partial t} \rangle = \frac{\partial \langle \mathbf{u} \rangle}{\partial t} \tag{4.5}$$

In order to express the ensemble average of the second term at left hand side it is necessary to recall the rule 3.7 of the ensemble average of a product. Indeed, expressing the velocity field as: $\mathbf{u} = \langle \mathbf{u} \rangle + \mathbf{u}'$, the convective term at left hand side is expressed by:

$$\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = (\langle \mathbf{u} \rangle + \mathbf{u}') \cdot \boldsymbol{\nabla} (\langle \mathbf{u} \rangle + \mathbf{u}') \tag{4.6}$$

Then, applying the ensemble average operation and accounting for the rule 3.7, the following expression is obtained for the ensemble average of the convective term:

$$\langle \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} \rangle = \langle \mathbf{u} \rangle \cdot \boldsymbol{\nabla} \langle \mathbf{u} \rangle + \langle \mathbf{u}' \cdot \boldsymbol{\nabla} \mathbf{u}' \rangle \tag{4.7}$$

Thanks to the fact that also the residual velocity field is solenoidal (equation 3.11), the second term at the right hand side of equation 4.7 can also be put in the form:

$$\langle \mathbf{u}' \cdot \boldsymbol{\nabla} \mathbf{u}' \rangle = \langle \boldsymbol{\nabla} \cdot (\mathbf{u}' \otimes \mathbf{u}') \rangle = \boldsymbol{\nabla} \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle \tag{4.8}$$

i.e. by adopting the expression of the tensor product of the velocity vector field with itself.
Finally, accounting for the previous results, the ensemble average momentum equation for an incompressible, viscous, Newtonian fluid or incompressible Reynolds average Navier-Stokes (RANS) equation takes the form:

$$\frac{\partial \langle \mathbf{u} \rangle}{\partial t} + \langle \mathbf{u} \rangle \cdot \boldsymbol{\nabla} \langle \mathbf{u} \rangle = -\frac{\boldsymbol{\nabla} \langle p \rangle}{\rho} + \mathbf{f}_b + \frac{\mu}{\rho} \boldsymbol{\nabla}^2 \langle \mathbf{u} \rangle - \boldsymbol{\nabla} \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle \tag{4.9}$$

The last term at right hand side of equation 4.9 comes from the decomposition of the velocity field into the average and residual velocity field and the ensemble average operation applied to the momentum equation: it is a new term that is equivalent to a stress. Indeed the tensor $\rho\langle \mathbf{u}' \otimes \mathbf{u}' \rangle$ is defined as the Reynolds stress tensor or apparent stress tensor. Its expression in terms of Cartesian components is:

$$\rho\langle \mathbf{u}' \otimes \mathbf{u}' \rangle = \rho \begin{pmatrix} \langle u_1'^2 \rangle & \langle u_1' u_2' \rangle & \langle u_1' u_3' \rangle \\ \langle u_2' u_1' \rangle & \langle u_2'^2 \rangle & \langle u_2' u_3' \rangle \\ \langle u_3' u_1' \rangle & \langle u_3' u_2' \rangle & \langle u_3'^2 \rangle \end{pmatrix} \tag{4.10}$$

The apparent stress tensor is due to inertia forces and acts mostly as an energy absorbing force. The latter statement can be better understood if the case of an incompressible fluid (e.g. water) in the gravitational field is considered. In this case the body force per unit mass is given by $\mathbf{f}_b = -g\boldsymbol{\nabla} x_3$ and can be considered in equation 4.9 together with the ensemble average pressure in the ensemble average piezometric head: $\langle \zeta \rangle = x_3 + \langle p \rangle / \rho g$.
Moreover, expressing the convective term at the left hand side of equation 4.9 as:

$$\langle \mathbf{u} \rangle \cdot \boldsymbol{\nabla} \langle \mathbf{u} \rangle = \frac{1}{2} \boldsymbol{\nabla} (\langle \mathbf{u} \rangle \cdot \langle \mathbf{u} \rangle) + \langle \boldsymbol{\omega} \rangle \times \langle \mathbf{u} \rangle \tag{4.11}$$

equation 4.9 can be rewritten as:

$$\frac{\partial \langle \mathbf{u} \rangle}{\partial t} + \langle \boldsymbol{\omega} \rangle \times \langle \mathbf{u} \rangle = -g \boldsymbol{\nabla} \mathcal{H} + \frac{\mu}{\rho} \boldsymbol{\nabla}^2 \langle \mathbf{u} \rangle - \boldsymbol{\nabla} \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle \tag{4.12}$$

Having defined the hydraulic head $\mathcal{H}$ of the ensemble averaged flow field as:

$$\mathcal{H} = \frac{\langle \mathbf{u} \rangle \cdot \langle \mathbf{u} \rangle}{2g} + \langle \zeta \rangle \tag{4.13}$$

Let us consider a steady flow, with negligible viscous stress, and project equation 4.12 along the direction of the velocity, represented by the unit vector $\mathbf{e}_s$. The following result is obtained:

$$g\frac{\partial \mathcal{H}}{\partial s} = -\boldsymbol{\nabla}\cdot\langle \mathbf{u}'\otimes\mathbf{u}'\rangle\cdot\mathbf{e}_s \tag{4.14}$$

In other words, due to the action of the turbulent stress, the hydraulic head $\mathcal{H}$ of the ensemble averaged flow field is not conserved also during steady motion of inviscid fluid (or very high Reynolds number).

4.2 Steady uniform turbulent flow

With reference to ensemble average values, a turbulent flow is defined to be steady if the local time derivative of an ensemble average of a whatever variable $\langle f\rangle$ vanishes identically: $\partial\langle f\rangle/\partial t = 0$, while it is defined to be uniform with respect to a given direction x_i if the spatial derivative of the ensemble average velocity vanishes identically: $\partial\langle f\rangle/\partial x_i = 0$.

4.2.1 Steady uniform turbulent 2D flow

Let us consider a 2D, steady, turbulent flow on a flat wall at $x_2 = 0$, where there is a prevailing velocity along the direction x_1: $\langle u_2\rangle = 0$ (figure 4.1).
The mass conservation equation takes the form:

$$\frac{\partial\langle u_1\rangle}{\partial x_1} = 0 \tag{4.15}$$

The flow is uniform with respect to the direction x_1. The velocity component $\langle u_1\rangle$ depends only on x_2. Therefore, the scalar components of the RANS equation assume the form:

$$-g\frac{\partial\langle\zeta\rangle}{\partial x_1}+\nu\frac{d^2\langle u_1\rangle}{dx_2^2}-\frac{\partial\langle(u_1')^2\rangle}{\partial x_1}-\frac{\partial\langle u_1'u_2'\rangle}{\partial x_2}=0$$
$$-g\frac{\partial\langle\zeta\rangle}{\partial x_2}-\frac{\partial\langle u_1'u_2'\rangle}{\partial x_1}-\frac{\partial\langle(u_2')^2\rangle}{\partial x_2}=0 \tag{4.16}$$

being $\langle\zeta\rangle$ the ensemble average piezometric head: $\langle\zeta\rangle = x_3 + \langle p\rangle/\rho g$.
It is usual to assume that also the Reynolds stress has a uniform behaviour, with respect to the direction x_1, so that the second equation 4.16 takes the form:

$$g\frac{\partial\langle\zeta\rangle}{\partial x_2}+\frac{\partial\langle(u_2')^2\rangle}{\partial x_2}=\frac{\partial}{\partial x_2}(g\langle\zeta\rangle+\langle(u_2')^2\rangle)=0 \tag{4.17}$$

whose solution is:

$$g\langle\zeta\rangle+\langle(u_2')^2\rangle=const \tag{4.18}$$

The constant value can be set to $g\zeta_w$, being ζ_w the piezometric head at the wall ($x_2 = 0$). It follows that the piezometric $\langle\zeta\rangle$ is not constant on the cross sections. Anyway, it is possible to assume $\langle\zeta\rangle \approx \zeta_w$ neglecting the Reynolds stress component $\langle(u_2')^2\rangle$ with reference to the other terms. Then the first equation 4.16 takes the form:

$$-g\frac{\partial\zeta_w}{\partial x_1}+\frac{d}{dx_2}\left(\nu\frac{d\langle u_1\rangle}{dx_2}-\langle u_1'u_2'\rangle\right)=0 \tag{4.19}$$

It is evident that the two terms at the left hand side of equation 4.19 are constant. Setting $-\partial\zeta_w/\partial x_1 = \mathrm{J}$ and $\nu\, d\langle u_1\rangle/dx_2 - \langle u_1'u_2'\rangle = -\tau/\rho$, equation 4.19 assumes the form:

$$\rho gJ-\frac{d\tau}{dx_2}=0 \tag{4.20}$$

τ and $-\tau$ are the tangential stresses that two fluid portions exchange with each other across an infinitesimal area with unit normal vector directed as x_2. Equation 4.20 can be solved immediately. The solution is:

$$\tau = \rho g J x_2 + const \tag{4.21}$$

According to equation 4.21 the total stress τ depends linearly on x_2.
Let us consider a second flat wall at $x_2 = d$. In order to determine the constant at the right hand side of equation 4.21, the condition that the total stress τ vanishes at $x_2 = d/2$ is imposed. This condition follows from the behaviour of the total stress: it is negative at $x_2 = 0$, it must have the same intensity and opposite sign at $x_2 = d$. It follows that at $x_2 = d/2$ the total stress must vanish. Then from equation 4.21, the value $const = -\rho g J\, d/2$ is obtained and the equation for the total stress τ takes the form:

$$\tau = -\rho g J \frac{d}{2}\left(1 - \frac{2\,x_2}{d}\right) \tag{4.22}$$

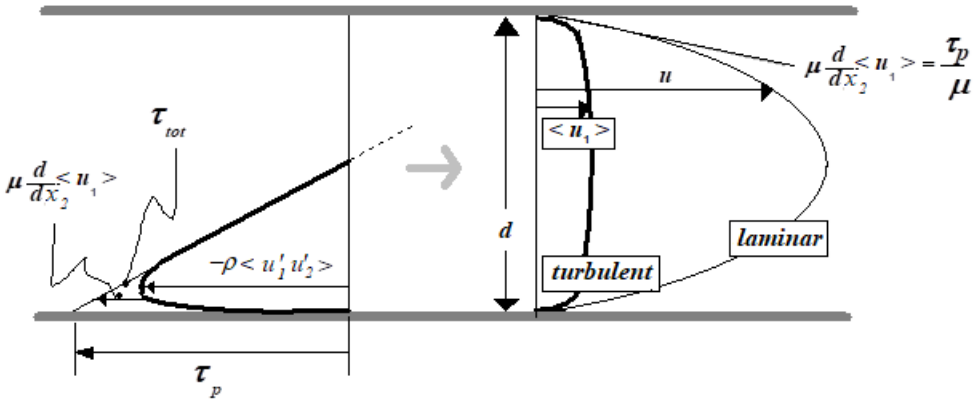

Figure 4.1: Uniform, steady, turbulent flow in a channel between two parallel walls.

The total stress τ is the sum of the viscous and the turbulent stress: the former is maximum at the wall, decreases rapidly with x_2 and becomes negligible (figure 4.1), the latter vanishes at the wall and increases rapidly with x_2 . At $x_2 = d/2$ the total stress vanishes. The total stress at the walls has the same intensity, but opposite sign due to the convention adopted on the stress: it is the stress action of the lower fluid layer on the upper one.
It is worth observing that two hypothetical flows, one laminar and the other turbulent, with the same piezometric slope and therefore the same τ, would

have the same slope as the velocity profile at the wall. Nevertheless, while for the laminar flow the velocity profile would assume a parabolic shape, for the turbulent flow the velocity profile would flatten quickly, determining a lower discharge.

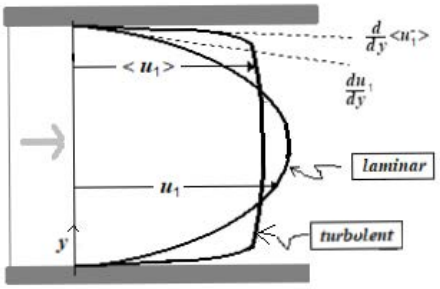

Figure 4.2: Comparison between laminar and turbulent motion.

Obviously two hypothetical flows with the same discharge (figure 4.2), one laminar and the other turbulent, should have two different values of τ and therefore of piezometric slope. In order to have the same discharge, the turbulent flow should have a greater τ and hence a greater piezometric slope. The comparison between the two motions is only hypothetical, because the flow, according to the Reynolds number, is either laminar or turbulent.

4.2.2 Velocity profile in the steady, uniform, turbulent flow

The profile of the ensemble average velocity in the steady, uniform, turbulent motion can be obtained by means of dimensional analysis.
In this case, at least two different regions must be distinguished: the wall region and the core region. The first is a region very close to the wall, while the second is a region close to the central part of the channel, where both turbulent and viscous stresses are gradually negligible. Within the wall region, both viscous and turbulent stresses are important and dominate each other: the former in the viscous sublayer, the latter in the inertial sublayer.
In the wall region the ensemble average velocity $\langle u_1 \rangle$, in the following indicated by u for the sake of simplicity, is assumed to be a function of the

distance x_2 from the wall, the stress at the wall τ_w, the viscosity and the density of the fluid:

$$f(u, x_2, \tau_w, \rho, \mu) = 0 \tag{4.23}$$

By applying the Buckingham theorem, the dimensional equation 4.23 can be reduced to the dimensionless equation:

$$\frac{u}{u_*} = f_*\left(\frac{\rho u x_2}{\mu}\right) \tag{4.24}$$

Being the friction velocity u_* defined as:

$$u_* = \sqrt{\frac{\tau_w}{\rho}} = \sqrt{gJ\frac{d}{2}} \tag{4.25}$$

The dimensionless parameter $\rho u x_2/\mu$ is often indicated with y_+. Then equation 4.24 becomes:

$$\frac{u}{u_*} = f_*(y_+) \tag{4.26}$$

The shape of the dimensionless function f_* can be obtained by applying equation 4.22, where the stress τ is expressed as the sum of the viscous and the Reynolds stress:

$$\nu\frac{du}{dx_2} - \langle u_1' u_2' \rangle = gJ\frac{d}{2}\left(1 - \frac{2x_2}{d}\right) \tag{4.27}$$

The wall region is analysed: $2x_2/d \ll 1$. Within this region the total stress can be assumed to be constant with respect to x_2. Moreover, in the viscous sublayer the turbulent stress is negligible, so that equation 4.27 takes the form:

$$\nu \frac{du}{dx_2} = gJ\frac{d}{2} = \frac{\tau_w}{\rho} \tag{4.28}$$

Equation 4.28 can be integrated, recalling the definition of the friction velocity u_* and of the dimensionless parameter y_+:

$$\frac{u}{u_*} = y_+ + const \tag{4.29}$$

The velocity profile in the viscous sublayer is linear with respect to y_+. The constant at the right hand side of equation 4.29 vanishes, due to the fact that at $y_+ = 0, u = 0$.
In the inertial sublayer, the viscous stress is negligible, so that equation 4.27 takes the form:

$$-\langle u_1' u_2' \rangle = u_*^2 \tag{4.30}$$

Equation 4.30 is useless, unless a closure equation is applied for the Reynolds stress[6].
However, in the inertial sublayer the viscosity affects u, as its values depend on the condition achieved at the edge of the viscous sublayer. But the variations of u with x_2 do not depend on the viscosity, i.e. the derivative of u does not depend on μ but only on ρ, x_2 and τ_w. Therefore, we have

$$f\left(\frac{du}{dx_2}, x_2, \tau_w, \rho\right) = 0 \tag{4.31}$$

By applying the Buckingham theorem, the dimensionless equation can be obtained:

$$f\left(\frac{du}{dx_2}\frac{x_2}{u_*}\right) = 0 \quad \Rightarrow \quad \frac{du}{dx_2}\frac{x_2}{u_*} = \frac{1}{\kappa} \tag{4.32}$$

[6] In this case the most common choice is the Boussinesq's hypothesis with the mixing length turbulence model

Whose solution is:

$$\frac{u}{u_*} = \frac{1}{\kappa}\ln x_2 + const \tag{4.33}$$

The constant κ is known as Von Karman constant.
The value of the constant at the right hand side of 4.33 can be obtained considering the velocity at a given distance from the wall. Let us assume that at $x_2 = \nu/u_*$ the velocity is equal to u_0:

$$const = \frac{u_0}{u_*} - \frac{1}{\kappa}\ln\frac{\nu}{u_*} \tag{4.34}$$

Then the velocity profile 4.34 takes the form:

$$\frac{u}{u_*} = \frac{u_0}{u_*} + \frac{1}{\kappa}\ln\frac{u_* x_2}{\nu} = \frac{u_0}{u_*} + \frac{1}{\kappa}\ln y_+ \tag{4.35}$$

Equation 4.35 is the well-known logarithmic velocity profile on a smooth wall. A huge number of results has confirmed it experimentally. The ratio u_0/u_* is usually obtained by interpolating the experimental results.
The logarithmic velocity profile can also be defined in the case of a rough surface: in this case, introducing the surface roughness ε, the logarithmic velocity profile takes the form:

$$\frac{u}{u_*} = \frac{u_0}{u_*} + \frac{1}{\kappa}\ln\frac{x_2}{\varepsilon} \tag{4.36}$$

Being in this case u_0 the velocity at $x_2 = \varepsilon$.
Figure 4.3 shows the velocity profile in the wall region on a smooth surface.

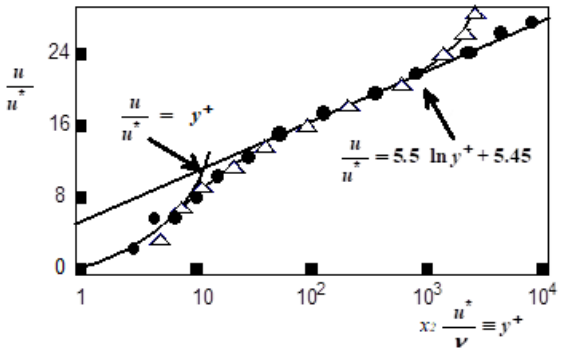

Figure 4.3: The velocity profile in the wall region.

The matching of the profiles in the viscous and inertial sublayer occurs conventionally at $y_+ = 11$. The velocity profile 4.29 in the viscous sublayer is valid for $0 \leq y_+ \leq 5$, while the velocity profile 4.35 in the inertial sublayer is valid for $y_+ \geq 30$. For $5 \leq y_+ \leq 30$ there is the so-called buffer layer.
The logarithmic velocity profile is valid for: $y_+ \leq 10^3$. For $y_+ > 10^3$, the experimental results show a meaningful deviation from the logarithmic law. In this region, where the assumption $2x_2/d \ll 1$ is no longer valid, a correction, defined the wake function, must be added to the logarithmic velocity profile, in order to represent accurately experimental results. From a technical point of view however, even the logarithmic law without corrections permits to obtain satisfying results.

4.2.3 Turbulent boundary layer on a flat plate

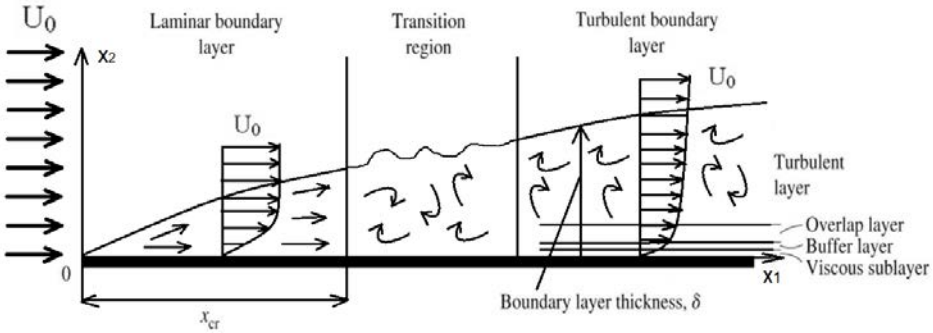

Figure 4.4: Development of the boundary layer and the 2D steady, uniform turbulent flow on a flat wall.

The boundary layer on a flat plate is laminar as long as the local Reynolds number:

$$Re_x = \frac{u_\infty x_1}{\nu} \tag{4.37}$$

is less than 10^6. As the local Reynolds number increases, the laminar to turbulent transition occurs (figure 4.4). For $Re_x > 10^8$ the flow is clearly turbulent (figure 4.5). u_∞ is the undisturbed velocity. One of the most important technical aspects in the turbulent boundary layer, as in the laminar boundary layer, is the determination of the tangential stress acted by the plate on the fluid and vice versa. To this aim, the knowledge of the velocity profile is fundamental. The latter is expressed by:

$$\frac{u - u_\infty}{u_*} = f\left(\frac{x_2}{\delta}\right) \tag{4.38}$$

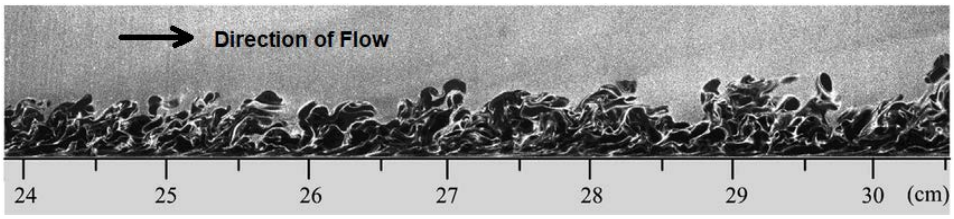

Figure 4.5: Turbulent boundary layer on a flat plate. Source: Impact of pure favorable pressure gradient on a supersonic flat-plate turbulent boundary layer, Qian-cheng Wang, Zhen-guo Wang, Yu-xin Zhao, Acta Astronautica, Volume 154, January 2019, Pages 67-73.

Equation 4.38 expresses the velocity profile as velocity defect (figure 4.6). δ is a length scale, characteristic for the wall region on the flat plate.
Assuming a logarithmic law, the velocity defect law 4.38 takes the form:

$$\frac{u - u_\infty}{u_*} = \frac{1}{\kappa} \ln \frac{x_2}{\delta} + B \tag{4.39}$$

Where B is the velocity defect at $x_2 = \delta$.
The velocity defect law is valid for $x_2 < 0.2\ \delta$. For $x_2 > 0.2\ \delta$ experimental data differ sensibly from the values given by equation 4.39 (figure 4.6).

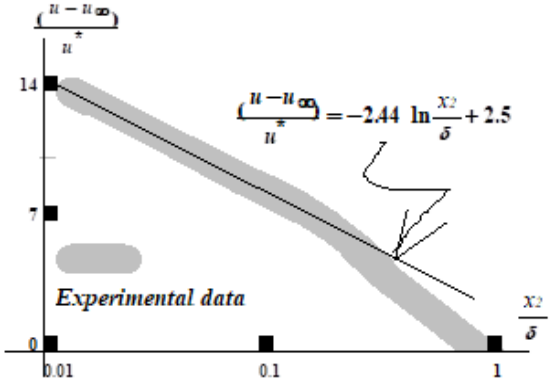

Figure 4.6: The logarithmic velocity defect.

Once the defect law is known, it is possible to determine the stress τ_w and then the drag force F on the surface σ. The latter is usually expressed by means of a drag coefficient c_D:

$$F = c_D \rho \frac{u_\infty^2}{2} \sigma \tag{4.40}$$

The drag coefficient depends on the Reynolds number. Typical behaviours for the laminar and turbulent flows are expressed respectively by:
$c_D = 1.33/\sqrt{Re}$ and $c_D = 3.91/Re^{2.58}$, shown in figure 4.7.

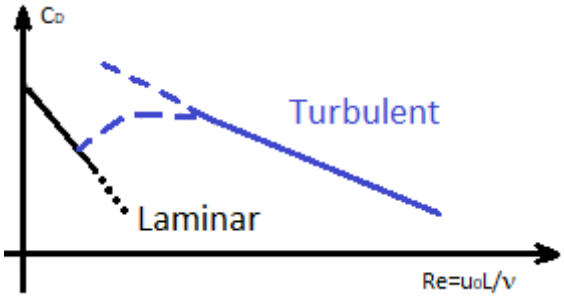

Figure 4.7: The drag coefficient.

CHAPTER 5 BALANCE EQUATIONS OF KINETIC ENERGY

5.1. Balance of the average kinetic energy

Let us consider the scalar product of the RANS equation 4.9 times the ensemble average velocity vector $\langle \mathbf{u} \rangle$ side by side and apply the ensemble average to the two sides:

$$\langle \mathbf{u} \rangle \cdot \left(\frac{\partial \langle \mathbf{u} \rangle}{\partial t} + \langle \mathbf{u} \rangle \cdot \boldsymbol{\nabla} \langle \mathbf{u} \rangle \right) = \langle \mathbf{u} \rangle \cdot \left(-\frac{\boldsymbol{\nabla} \langle p \rangle}{\rho} + \mathbf{f}_b + \frac{\mu}{\rho} \boldsymbol{\nabla}^2 \langle \mathbf{u} \rangle - \boldsymbol{\nabla} \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle \right) \quad (5.1)$$

The left hand side of equation 5.1 can be rewritten as:

$$\langle \mathbf{u} \rangle \cdot \left(\frac{\partial \langle \mathbf{u} \rangle}{\partial t} + \langle \mathbf{u} \rangle \cdot \boldsymbol{\nabla} \langle \mathbf{u} \rangle \right) = \frac{\partial}{\partial t} \left(\frac{\langle \mathbf{u} \rangle \cdot \langle \mathbf{u} \rangle}{2} \right) + \langle \mathbf{u} \rangle \cdot \boldsymbol{\nabla} \left(\frac{\langle \mathbf{u} \rangle \cdot \langle \mathbf{u} \rangle}{2} \right) \quad (5.2)$$

i.e. is the material derivative of the average kinetic energy per unit mass K ($K = \langle \mathbf{u} \rangle \cdot \langle \mathbf{u} \rangle / 2$), defined in terms of the ensemble average velocity field. The last term at right hand side of equation 5.2 can be put in the form

$$\langle \mathbf{u} \rangle \cdot \boldsymbol{\nabla} \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle = \boldsymbol{\nabla} \cdot (\langle \mathbf{u}' \otimes \mathbf{u}' \rangle \langle \mathbf{u} \rangle) - \langle \mathbf{u}' \otimes \mathbf{u}' \rangle : \boldsymbol{\nabla} \langle \mathbf{u} \rangle \quad (5.3)$$

so that equation 5.2 takes the form:

$$\frac{\partial K}{\partial t} + \langle \mathbf{u} \rangle \cdot \boldsymbol{\nabla} K = -\frac{\boldsymbol{\nabla} \cdot (\langle p \rangle \langle \mathbf{u} \rangle)}{\rho} + \langle \mathbf{u} \rangle \cdot \mathbf{f}_b + \frac{\mu}{\rho} \langle \mathbf{u} \rangle \cdot \boldsymbol{\nabla}^2 \langle \mathbf{u} \rangle - \boldsymbol{\nabla} \cdot (\langle \mathbf{u}' \otimes \mathbf{u}' \rangle \langle \mathbf{u} \rangle) + \langle \mathbf{u}' \otimes \mathbf{u}' \rangle : \boldsymbol{\nabla} \langle \mathbf{u} \rangle \quad (5.4)$$

The first two terms at right hand side are the contributions to the kinetic energy given by the pressure and the body force. The third term at right hand side is the energy dissipated per unit time by the viscous stresses, defined in terms of the ensemble average velocity field.

Finally, the last two terms at the right hand side of equation 5.4 are expressed in terms of the Reynolds stress. The first is the energy per unit time produced by the Reynolds stress and is generally negligible; the second is the energy per unit time generated by the Reynolds stress needed to deform the average flow field. This term, known as turbulent deformation work, is not negligible and is a sink term for the kinetic energy. The turbulent deformation work takes energy from the ensemble averaged flow and transfers it to the fluctuating flow.
The expression of the turbulent deformation work in terms of Cartesian components is given by:

$$\langle \mathbf{u}' \otimes \mathbf{u}' \rangle : \boldsymbol{\nabla} \langle \mathbf{u} \rangle = \langle u_i' u_j' \rangle \frac{\partial \langle u_i \rangle}{\partial x_j} \tag{5.5}$$

The deformation of the flow field results mainly in tilting and stretching of the vortex filaments. The former bends the vortex filament, increasing the three-dimensionality of motion, the latter stretches the vortex filament, reducing its dimension and increasing the vorticity at the same time. This process is performed, until the dissipation becomes important, absorbing the whole energy extracted by the deformation work.

5.2 Balance of the turbulent kinetic energy

Let us consider the scalar product of the Navier-Stokes equation for an incompressible fluid times the fluctuating velocity vector field $\mathbf{u}'$:

$$\mathbf{u}' \cdot \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} \right) = \mathbf{u}' \cdot \left(-\frac{\boldsymbol{\nabla} p}{\rho} + \mathbf{f}_b + \frac{\mu}{\rho} \boldsymbol{\nabla}^2 \mathbf{u} \right) \tag{5.6}$$

Let us express the velocity field as the sum of the ensemble average and the residual velocity field and then perform the ensemble average operation. The result is obtained after a cumbersome series of steps, which are shown in detail in the following.

From the first term at left hand side of equation 5.6 one obtains:

$$\langle \mathbf{u}' \cdot \frac{\partial \mathbf{u}}{\partial t} \rangle = \langle \mathbf{u}' \cdot \frac{\partial}{\partial t} (\langle \mathbf{u} \rangle + \mathbf{u}') \rangle = \frac{\partial}{\partial t} \langle \frac{\mathbf{u}' \cdot \mathbf{u}'}{\mathbf{2}} \rangle \tag{5.7}$$

Introducing the turbulent kinetic energy K' per unit mass:

$$K' = \frac{\mathbf{u}' \cdot \mathbf{u}'}{2} \tag{5.8}$$

the term 5.7 can be rewritten as:

$$\langle \mathbf{u}' \cdot \frac{\partial \mathbf{u}}{\partial t} \rangle = \frac{\partial \langle K' \rangle}{\partial t} \tag{5.9}$$

From the second term at left hand side of equation 5.6 one obtains:

$$\langle \mathbf{u}' \cdot \mathbf{u} \cdot \nabla \mathbf{u} \rangle = \langle \mathbf{u}' \cdot \langle \mathbf{u} \rangle \cdot \nabla \langle \mathbf{u} \rangle \rangle + \langle \mathbf{u}' \cdot \mathbf{u}' \cdot \nabla \langle \mathbf{u} \rangle \rangle + \langle \mathbf{u}' \cdot \langle \mathbf{u} \rangle \cdot \nabla \mathbf{u}' \rangle + \langle \mathbf{u}' \cdot \mathbf{u}' \cdot \nabla \mathbf{u}' \rangle \tag{5.10}$$

which can be put in the form:

$$\langle \mathbf{u}' \cdot \mathbf{u} \cdot \nabla \mathbf{u} \rangle = \langle \mathbf{u}' \otimes \mathbf{u}' \rangle : \nabla \langle \mathbf{u} \rangle + \langle \langle \mathbf{u} \rangle \cdot \nabla K' \rangle + \langle \mathbf{u}' \cdot \nabla K' \rangle \tag{5.11}$$

Summing the second and third term of equation 5.11, the latter takes the form:

$$\langle \mathbf{u}' \cdot \mathbf{u} \cdot \nabla \mathbf{u} \rangle = \langle \mathbf{u}' \otimes \mathbf{u}' \rangle : \nabla \langle \mathbf{u} \rangle + \langle \mathbf{u} \cdot \nabla K' \rangle \tag{5.12}$$

The first term at the right hand side of equation 5.6 side gives:

$$\langle \mathbf{u}' \cdot \frac{\nabla p}{\rho} \rangle = \langle \mathbf{u}' \cdot \frac{\nabla \langle p \rangle}{\rho} \rangle + \langle \mathbf{u}' \cdot \frac{\nabla p'}{\rho} \rangle = \frac{1}{\rho} \nabla \cdot \langle p' \mathbf{u}' \rangle \tag{5.13}$$

While the second term at right hand side vanishes identically:

$$\langle \mathbf{u}' \cdot \mathbf{f}_b \rangle = 0 \tag{5.14}$$

Finally, the last term at right hand side gives:

$$\frac{\mu}{\rho} \langle \mathbf{u}' \cdot \boldsymbol{\nabla}^2 \mathbf{u} \rangle = \frac{2\mu}{\rho} \boldsymbol{\nabla} \cdot \langle \mathbb{D}\, \mathbf{u}' \rangle - \frac{2\mu}{\rho} \langle \mathbb{D}' : \mathbb{D} \rangle \tag{5.15}$$

expressing as usual the velocity field as the sum of the ensemble average and the residual field and performing the ensemble average operation, equation 5.15 takes the form:

$$\frac{2\mu}{\rho} \boldsymbol{\nabla} \cdot \langle \mathbb{D}\, \mathbf{u}' \rangle - \frac{2\mu}{\rho} \langle \mathbb{D}' : \mathbb{D} \rangle = \frac{2\mu}{\rho} \boldsymbol{\nabla} \cdot \langle \mathbb{D}'\, \mathbf{u}' \rangle - \frac{2\mu}{\rho} \langle \mathbb{D}' : \mathbb{D}' \rangle \tag{5.16}$$

Being $\mathbb{D}'$ the symmetric part of the second order tensor $\boldsymbol{\nabla}\mathbf{u}'$. Collecting the previous results, the balance of the turbulent kinetic energy is obtained:

$$\frac{\partial \langle K\prime \rangle}{\partial t} + \langle \mathbf{u} \cdot \boldsymbol{\nabla} K' \rangle = -\frac{1}{\rho} \boldsymbol{\nabla} \cdot (\langle p' \mathbf{u}' \rangle - 2\mu \langle \mathbb{D}' \mathbf{u}' \rangle) - \langle \mathbf{u}' \otimes \mathbf{u}' \rangle : \boldsymbol{\nabla} \langle \mathbf{u} \rangle - \frac{2\mu}{\rho} \langle \mathbb{D}' : \mathbb{D}' \rangle \tag{5.17}$$

The meaning of the terms of equation 5.17 is explained in the following. The second term at left hand side expresses the transport of turbulent kinetic energy due to the ensemble average and residual velocity field. It is a pure convection term and does not account for dissipation at all. The first term at right hand side expresses energy transferred from point to point, due to pressure and viscous stresses. The term $2\mu \boldsymbol{\nabla} \cdot \langle \mathbb{D}' \mathbf{u}' \rangle$ accounts mainly for the diffusion of turbulent kinetic energy due to the residual velocity. Indeed, it can be rewritten as:

$$2\mu \boldsymbol{\nabla} \cdot \langle \mathbb{D}' \mathbf{u}' \rangle = 2\mu \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \langle K' \rangle + \mu \boldsymbol{\nabla} \cdot \langle \boldsymbol{\omega}' \times \mathbf{u}' \rangle = 2\mu \boldsymbol{\nabla}^2 \langle K' \rangle + \mu \boldsymbol{\nabla} \cdot \langle \boldsymbol{\omega}' \times \mathbf{u}' \rangle \tag{5.18}$$

The third term at right hand side of equation 5.17 appears, with opposite sign, in the kinetic energy equation 5.4 too: it represents the work done by the Reynolds stresses to deform the fluid and is generally a negative contribution in the kinetic energy equation 5.4, while it is a positive contribution in the turbulent kinetic equation 5.17. In other words this term represents the energy transferred from the ensemble average motion to feed the residual motion, i.e. the energy extracted from the kinetic energy relative to the ensemble average motion which increases the turbulent kinetic energy. This term, thanks to the symmetry of the Reynolds stress second order tensor, can be expressed by means of the symmetric part of the deformation velocity second order tensor $\langle \mathbb{D} \rangle$:

$$\langle \mathbf{u}' \otimes \mathbf{u}' \rangle : \nabla \langle \mathbf{u} \rangle = \langle \mathbf{u}' \otimes \mathbf{u}' \rangle : \langle \mathbb{D} \rangle \tag{5.19}$$

The fourth term at right hand side of equation 5.17 represents the dissipation of turbulent kinetic energy due to the residual motion. The energy flux can be represented as follows: mechanical energy is transferred from the ensemble average flow field to the residual flow field thanks the work done by the Reynolds stresses and is eventually dissipated by the viscous stresses caused by the residual flow field.

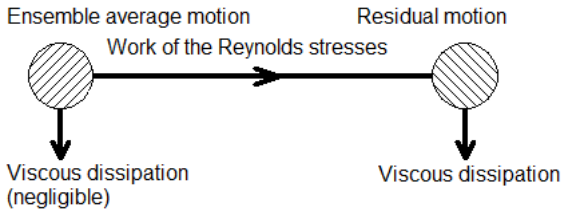

Figure 5.1: Synthetic diagram of the energy flux between the ensemble average and the residual flow field.

5.3. The length scale of the fluctuating motion

The term $2\mu\langle\mathbb{D}'\colon\mathbb{D}'\rangle/\rho$ dissipates efficiently the turbulent kinetic energy K' because the components of the symmetric deformation velocity second order tensor $\mathbb{D}'$ are large enough. The order of magnitude of the latter can be expressed as:

$$D'_{ij} \approx \frac{\sqrt{\langle K'\rangle}}{\boldsymbol{\ell}} \tag{5.20}$$

Being $\boldsymbol{\ell}$ the length scale over which variations of residual velocity of order of magnitude equal to $\sqrt{\langle K'\rangle}$ occur. Thanks to formula 5.20, the order of magnitude of the dissipation of turbulent kinetic is expressed as:

$$\frac{\mu}{\rho}\langle\mathbb{D}'\colon\mathbb{D}'\rangle \approx \frac{\mu}{\rho}\frac{\langle K'\rangle}{\boldsymbol{\ell}^2} \tag{5.21}$$

The order of magnitude of the term expressing the energy transfer from the kinetic to the turbulent kinetic energy: $\langle\mathbf{u}'\otimes\mathbf{u}'\rangle\colon\langle\mathbb{D}\rangle$ is given by:

$$\langle\mathbf{u}'\otimes\mathbf{u}'\rangle\colon\langle\mathbb{D}\rangle \approx \langle K'\rangle\frac{U}{\mathcal{L}} \tag{5.22}$$

being $U,\mathcal{L}$ the velocity and length scales of the ensemble average flow respectively, which determine the order of magnitude of $\langle\mathbb{D}\rangle$ as: $\langle\mathbb{D}\rangle \approx U/\mathcal{L}$. At equilibrium, third and fourth term at right hand side of equation 5.17 have same order of magnitude: this means that the energy transferred from the ensemble average length scales is dissipated at residual motion length scales:

$$\frac{\mu}{\rho}\frac{\langle K'\rangle}{\boldsymbol{\ell}^2} \approx \langle K'\rangle\frac{U}{\mathcal{L}} \tag{5.23}$$

The ratio $\boldsymbol{\ell}/\mathcal{L}$ is obtained from equation 4.37 as:

$$\frac{\boldsymbol{\ell}}{\mathcal{L}} \approx \sqrt{\frac{\mu}{\rho U \mathcal{L}}} = \frac{1}{\sqrt{Re}} \tag{5.24}$$

It is interesting to observe that, while $\mathcal{L}$ is characterized by the geometry of the fluid domain, $\boldsymbol{\ell}$ is determined by the considered flow in terms of the Reynolds number: the larger the Reynolds number, the smaller the length scale $\boldsymbol{\ell}$ of the fluctuating motion.
The ratio $\boldsymbol{\ell}/\mathcal{L}$ gives an idea of how many grid points should be taken on the length scale $\mathcal{L}$ along a given direction to give a sufficiently accurate description of the turbulent motion. Indeed, if the smallest lengthscale to be represented is $\boldsymbol{\ell}$, the number of grid points needed to give a sufficiently accurate description of the flow within the volume $\mathcal{L}^3$ is given by:

$$\left(\frac{\mathcal{L}}{\boldsymbol{\ell}}\right)^3 \approx Re^{\frac{3}{2}} \tag{5.25}$$

Due to the high values of the Reynolds number characterizing turbulent flows, the number of grid points necessary to give a sufficiently accurate description of the flow can become huge.

5.4. Spectral distribution of the kinetic energy in a turbulent flow

5.4.1. Spatial Fourier modes and their energy balance

Continuous vector or scalar functions of spatial variables can be expressed as sums of series of spatial harmonic functions. Consider a parallelepiped fluid domain, with sides parallel to the three orthogonal spatial directions. The fluid velocity can be expressed as:

$$\mathbf{u}(\mathbf{x}) = \sum_{\substack{k_1=-\infty \\ k_2=-\infty \\ k_3=-\infty}}^{\infty} \mathbf{u_k} e^{I\mathbf{k}\cdot\mathbf{x}} \tag{5.26}$$

In which, *I* is the imaginary unit, $\mathbf{u_k}$ is the Fourier coefficient relative to the wave vector $\mathbf{k} = k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + k_3\mathbf{e}_3$. The sum at right hand side of equation (5.26) can be also expressed more concisely as:

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{u_k} e^{I\mathbf{k}\cdot\mathbf{x}}$$

The Cartesian components of the wave vector k_1, k_2, k_3 are defined as:

$k_1 = 2\pi l/L_1\,, k_2 = 2\pi m/L_2\,, k_3 = 2\pi n/L_3\,,$ being l, m, n integers and L_1, L_2, L_3 the sides of the parallelepiped fluid domain. The Fourier coefficient $\mathbf{u_k}$ depends only on time and is defined by:

$$\mathbf{u_k} = \frac{\mathbf{1}}{L_1 L_2 L_3} \int_{\mathbf{0}}^{L_1} \int_{\mathbf{0}}^{L_2} \int_{\mathbf{0}}^{L_3} \mathbf{u} e^{-I\mathbf{k}\cdot\mathbf{x}} dx_1 dx_2 dx_3 \tag{5.27}$$[7]

Let's consider the Navier-Stokes equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -g\boldsymbol{\nabla}\zeta + \frac{\mu}{\rho}\boldsymbol{\nabla}^2\mathbf{u} \tag{5.28}$$

in which the piezometric height $\zeta$is defined as: $\zeta = x_3 + p/\rho g$.

[7] Since the series is bounded, as $|\mathbf{k}|$ increases the amplitudes of the modes decrease. The term calculated with $\mathbf{k}$=0, $\mathbf{u}_0$, is real, but for the convenience of the agebric expansion it is expressed as the sum of a complex number and its conjugate whose imaginary part is constant.

By substituting (5.26) and a similar expression for the piezometric height into the Navier-Stokes equation we obtain[8]:

$$\sum_{\mathbf{k}} \left(\frac{d\mathbf{u_k}}{dt} + gI\mathbf{k}\zeta_{\mathbf{k}} + \frac{\mu}{\rho}|\mathbf{k}|^2\mathbf{u_k} \right) e^{I\mathbf{k}\cdot\mathbf{x}} + \sum_{\mathbf{l}} \sum_{\mathbf{m}} (I\mathbf{m}\cdot\mathbf{u_l})\mathbf{u_m} e^{I(\mathbf{l}+\mathbf{m})\cdot\mathbf{x}} = 0 \qquad (5.29)$$

From equation (5.29) it is possible to obtain the balance of both the averaged and the fluctuating kinetic energy contribution given by the Fourier harmonics relative to the wave vector $\mathbf{k}$.

As for the averaged kinetic energy, we express the scalar product of equation (5.29) times $\langle \mathbf{u_n} \rangle e^{I\mathbf{n}\cdot\mathbf{x}}$ and integrate term by term over the entire fluid domain with respect to the spatial variables. After straightforward calculations we get:

$$\frac{d}{dt}\frac{\mathbf{u_{-n}}\cdot\langle \mathbf{u_n} \rangle}{2} - gI\mathbf{n}\zeta_{-\mathbf{n}} \cdot \langle \mathbf{u_n} \rangle + \frac{\mu}{\rho}|\mathbf{n}|^2\mathbf{u_{-n}} \cdot \langle \mathbf{u_n} \rangle + \sum (I\mathbf{m}\cdot\mathbf{u_l})\mathbf{u_m} \cdot \langle \mathbf{u_n} \rangle = 0 \qquad (5.30)$$

For any given wave vector $\mathbf{n}$, the sum at left hand side of equation (5.30) accounts only for the terms whose wave vectors satisfy the condition:

$\mathbf{l} + \mathbf{m} + \mathbf{n} = \mathbf{0}$.

By expressing the Fourier coefficients as the sum of an averaged and residual part and applying the ensemble average operation, we obtain:

$$\frac{d}{dt}\frac{|\langle \mathbf{u_n} \rangle|^2}{2} - gI\mathbf{n}\langle \zeta_{-\mathbf{n}} \rangle \cdot \langle \mathbf{u_n} \rangle + \frac{\mu}{\rho}|\mathbf{n}|^2|\langle \mathbf{u_n} \rangle|^2 + I \sum \mathbf{m} \cdot (\langle \mathbf{u}_l \rangle \langle \mathbf{u_m} \rangle + \langle \mathbf{u'}_\mathbf{l} \mathbf{u'}_\mathbf{m} \rangle) \cdot \langle \mathbf{u_n} \rangle = 0 \qquad (5.31)$$

[8] The following operations have been applied:

$$\nabla\zeta_{\mathbf{k}} e^{I\mathbf{k}\cdot\mathbf{x}} = I\mathbf{k}\zeta_{\mathbf{k}} e^{I\mathbf{k}\cdot\mathbf{x}}$$
$$\nabla\mathbf{u_k} e^{I\mathbf{k}\cdot\mathbf{x}} = (\mathbf{u_k} \otimes I\mathbf{k}) e^{I\mathbf{k}\cdot\mathbf{x}}$$
$$\nabla^2\mathbf{u_k} e^{-I\mathbf{k}\cdot\mathbf{x}} = -|\mathbf{k}|^2\mathbf{u_k}\, e^{I\mathbf{k}\cdot\mathbf{x}}$$
$$\mathbf{u_l} e^{I\mathbf{l}\cdot\mathbf{x}} \nabla\mathbf{u_m} e^{I\mathbf{m}\cdot\mathbf{x}} = \mathbf{u_l}(\mathbf{u_m} \otimes I\mathbf{m}) e^{I(\mathbf{l}+\mathbf{m})\cdot\mathbf{x}}$$

where the symbol $\otimes$ represents the usual tensor product.

As for the fluctuating kinetic energy, we express the scalar product of equation (5.29) times $\mathbf{u'_n}e^{I\mathbf{n}\cdot\mathbf{x}}$ and integrate term by term over the entire fluid domain with respect to the spatial variables. We get:

$$\frac{d}{dt}\frac{\mathbf{u_{-n}}\cdot\mathbf{u'_n}}{2}+gI\mathbf{n}\zeta_{-\mathbf{n}}\cdot\mathbf{u'_n}+\frac{\mu}{\rho}|\mathbf{n}|^2\mathbf{u_{-n}}\cdot\mathbf{u'_n}+\sum(I\mathbf{m}\cdot\mathbf{u_l})\mathbf{u_m}\cdot\mathbf{u'_n}=0 \tag{5.32}$$

By expressing the Fourier coefficients as the sum of an averaged and residual part and applying the ensemble average operation, after straightforward calculations we obtain:

$$\frac{d}{dt}\frac{\langle|\mathbf{u'_n}|^2\rangle}{2}+gI\mathbf{n}\cdot\langle\zeta'_{-\mathbf{n}}\mathbf{u'_n}\rangle+\frac{\mu}{\rho}|\mathbf{n}|^2\langle|\mathbf{u'_n}|^2\rangle+$$
$$+I\sum\left(\mathbf{m}\cdot\langle\mathbf{u_l}\rangle\langle\mathbf{u'_m}\mathbf{u'_n}\rangle+m_j\langle u'_j\mathbf{l}\mathbf{u'_n}\rangle\cdot\langle\mathbf{u_m}\rangle+\mathbf{m}\cdot\langle\mathbf{u'_l}\mathbf{u'_m}\cdot\mathbf{u'_n}\rangle\right)=0 \tag{5.33}$$

For any given wave vector $\mathbf{n}$, the sum at left hand side of equation (5.33) accounts only for the terms whose wave vectors satisfy the condition:
$\mathbf{l}+\mathbf{m}+\mathbf{n}=\mathbf{0}$.
Note that in stationary and uniform motion the first term at left hand side of equation (5.33) vanishes. While, the second term is different from zero and transfers power to the fluctuating modes.

5.4.2 Considerations on the energy balances of Fourier modes

Considering equations (5.31) and (5.33), it is evident that:

- the power transfer among Fourier modes, i.e. the interaction among different modes, occurs between triples of modes, whose wave numbers are linked by the relationship: $\mathbf{l}+\mathbf{m}+\mathbf{n}=\mathbf{0}$.
- The average kinetic energy is fed by the average piezometric gradient, negligibly dissipated by the viscous term, and redistributed on the fluctuating motions thanks to the convective term, the last term at left hand side of equation (5.31).

- The same term is found in equation (5.33). The relative power is balanced by the viscous dissipation term, the third term at right hand side of equation (5.33), and redistributed among other modes through the other addends of the sum
- At a given Reynolds number, the equilibrium wave vector is defined as that wave vector for which the power dissipated by viscosity is equal to the power produced by the average piezometric gradient. As the Reynolds number increases, i.e. as the viscosity decreases, the modulus of the equilibrium wave vector increases and so does the spatial frequency, thus determining a "fine-grained" turbulence.
- The fluctuating modes with a wave number different from the equilibrium one transfer power to the other modes through the convective term. The modes corresponding to large spatial scales (wave vectors with small modulus) give more power to those corresponding to small spatial scales (wave vectors with large modulus) than that received from these latter. This fact determines a general transfer of power from the lower wavenumber modes to those with higher wavenumber (energy cascade).
- At the low wave number modes, the power supplied from external sources is dominant and, apart from a negligible dissipation, is almost completely transferred to the modes with higher wave number.

  At the high wavenumber modes, on the contrary, viscous dissipation prevails on the power supplied from external sources. The high wavenumber modes are indeed fed by the power received by the lower wavenumber modes.

  Finally, in the intermediate wave number modes, the viscous dissipation and the transfer from the outside are balanced and the transfer to and from the fluctuating modes is negligible.

5.5. Three-dimensional spectrum of fluctuating kinetic energy of the homogeneous and isotropic turbulent flow (Kolmogorov theory)

The ensemble average turbulent kinetic energy $\langle K' \rangle$ can be represented in terms of wavelengths, considering all the possible wavenumber vectors. Anyway, assuming that the turbulent flow characteristics are both homogeneous and isotropic, i.e. they neither do depend on position nor on direction, the ensemble average turbulent kinetic energy $\langle K' \rangle$ can be represented in terms of the modulus k of the wavenumber vector ($k = |\mathbf{k}|$) as:

$$\langle K' \rangle = \int_0^\infty E(k)dk \tag{5.34}$$

being $E(k)dk$ the contribution to the ensemble average turbulent kinetic energy $\langle K' \rangle$ due to the wavelengths, whose wavenumbers belong to the range $k, k + dk$. The function $E(k)$ is the spectral density of ensemble average turbulent kinetic energy. The function $E(k)$ describes how the ensemble average turbulent kinetic energy $\langle K' \rangle$ is distributed among the wavelengths. The shape of $E(k)$ depends on the considered turbulent flow for small k, i.e. for large length scales, representative for the spatial dimension of the fluid domain (e.g. the diameter of a pipe, the depth of a channel, etc.), while it assumes a unique behaviour for large k, i.e. for small length scale, representative for the fluctuating motion. In other words, the behaviour at large k is universal, does not depend on the considered turbulent flow, because at the smallest length scales the turbulent behaviour is universal and consists of the dissipation of the energy transferred from the large length scales. The Russian scientist A. N. Kolmogorov first acknowledged such a behaviour in turbulent flows and obtained the expression for $E(k)$. He considered the turbulence homogeneous and isotropic and postulated that energy transfer between two very different length scales is not probable. He therefore divided the values of k into three intervals: the energy production interval at small values of k, the inertial

interval at intermediate wavenumber, where energy is transferred from low to high wavenumber with negligible production and dissipation, and the dissipative range at the largest wave numbers, where energy is dissipated by viscosity. Given the hypothesis of Kolmogorov, these ranges are independent from each other. Moreover, the inertial and dissipative ranges are defined as universal equilibrium ranges, because they depend only on the energy produced and transferred from the largest length scales. In these ranges, the function $E(k)$ has to depend only on the dissipated energy per unit mass ε ($\varepsilon = 2\nu\langle \mathbb{D}' : \mathbb{D}' \rangle$), equal to the produced energy, and on the viscosity ν. The Kolmogorov length and velocity scales are defined in terms of ε, ν as follows:

$$\lambda_K = \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}}, \qquad u_K = (\nu\varepsilon)^{\frac{1}{4}} \tag{5.35}$$

In the universal equilibrium range a relation among E, k, ε, ν exists:

$$f(E, k, \varepsilon, \nu) = 0 \tag{5.36}$$

The physical dimensions of $E(k)$ are a velocity square times a length. Then by applying the Buckingham theorem and accounting for the Kolmogorov length and velocity scales 5.35, the relation 5.36 can be expressed as:

$$\frac{E(k)}{u_K^2 \lambda_K} = f^*(k\lambda_K) \tag{5.37}$$

Equation 5.36 can be simplified for $Re \to \infty$ i.e. for $\nu \to 0$. Within this limit the function f does not depend on ν:

$$f(E, k, \varepsilon) = 0 \tag{5.38}$$

Hence the famous Kolmogorov power law can be obtained:

$$\frac{E}{\varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}} = const \tag{5.39}$$

The validity of the Kolmogorov power law 5.39 can be extended to flows

with finite values of the Reynolds number, for wavenumbers k belonging to the inertial range, i.e. greater than those of the energy production range ($k \gg 1/L$) but smaller than those of the dissipation range ($k \ll 1/\lambda_K$), as shown in figure 5.2. The wavenumber $k \approx 1/\lambda_K$ can be considered as the start of the dissipation range. For increasing Reynolds number the Kolmogorov length scale λ_K becomes smaller and smaller and the inertial range expands: consequently the validity range of the Kolmogorov power law expands too.

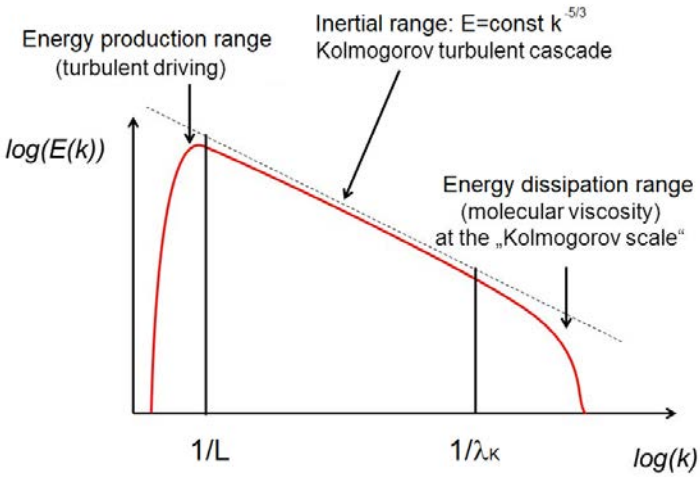

Figure 5.2: The energy spectrum E(k) and the Kolmogorov power law for finite Re. For Re tending to infinite, $1/\lambda_K$ tends to infinite too and the range of validity of Kolmogorov power law extends to infinite.

Kolmogorov law has been tested for very high Re numbers and its validity has been experimentally confirmed with good approximation. It should be noted that, after some theoretical criticism, Kolmogorov made some changes to his approach. In any case, Kolmogorov's law of power has been very successful and today is considered reliable.

CHAPTER 6 TURBULENCE MODELS

6.1. The Direct Numerical Simulation (DNS)

Turbulent flows of viscous Newtonian fluid are governed by the Navier Stokes equation. In other words, a given turbulent flow of a viscous Newtonian fluid is the solution of the Navier-Stokes equation with suitable boundary and initial conditions. In principle, one should be able to get the mathematical description of the considered turbulent flow by solving the Navier-Stokes equation. The problem is that, except for few laminar flow cases, the solutions of the Navier-Stokes equation cannot be determined analytically but only by means of numerical methods, which often imply preliminary treatments on the equation itself. The Direct Numerical Simulation (DNS) is the numerical approach to the investigation on turbulent flows which does not require any assumption on the Navier-Stokes equation and solve it as is.

The main difficulty of the DNS depends on the high number of computational cells needed to represent adequately the considered turbulent motion. Indeed this number increases sensibly with the Reynolds number. In order to obtain a meaningful and accurate description of the flow, the order of magnitude of the spatial extension of the smallest subdomain, i.e. the computational cell, has to be at least equal to the Kolmogorov length scale λ_K. The latter is given by the equation 4.55 in terms of the viscosity ν and the dissipated energy per unit mass ε, whose order of magnitude, assuming the hypothesis of equilibrium turbulence, is the same as the kinetic energy per unit mass and time, extracted from the ensemble average flow field. In other words the order of magnitude of the kinetic energy per unit mass is: U^2 and is extracted from the ensemble average motion during the time interval $\mathcal{L}/U$, so the order of magnitude of the kinetic energy per unit mass and time is given by: $U^3/\mathcal{L}$. Then the Kolmogorov length scale can be expressed as:

$$\lambda_K = \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}} = \left(\frac{\mathcal{L}\nu^3}{U^3}\right)^{\frac{1}{4}} \tag{6.1}$$

and consequently the ratio $\mathcal{L}/\lambda_K$ can be expressed in terms of the Reynolds number as:

$$\frac{\mathcal{L}}{\lambda_K} = \left(\frac{U\mathcal{L}}{\nu}\right)^{\frac{3}{4}} = Re^{\frac{3}{4}} \tag{6.2}$$

Of course the ratio $\mathcal{L}/\lambda_K$ gives the number of spatial intervals of amplitude λ_K contained in the interval $\mathcal{L}$, the length scale of the average flow. Then the order of magnitude of the volume of the spatial domain of the flow is given by $\mathcal{L}^3$ and the number of subdomains to be considered is given by:

$$\left(\frac{\mathcal{L}}{\lambda_K}\right)^3 = Re^{\frac{9}{4}} \tag{6.3}$$

The number of subdomains may become huge. Let us think of a turbulent flow characterized by $Re = 10^4$, which is not so high: it needs a number of points equal to 10^9 which is challenging, mainly for the data post-processing. Then the realization of the DNS of a given turbulent flow is possible or not depending on the available computational resources and requires the use of supercomputers.

Similar conclusions can be obtained if, instead of λ_K, a different length scale, characteristic for the fluctuating motion, is assumed. In fact, (5.25) was obtained by considering $\boldsymbol{\ell}$ the length scale over which variations of fluctuating velocity of order of magnitude equal to $\sqrt{\langle K' \rangle}$ occur: it also shows an exponentially increasing function of Re with exponent equal to 3/2.

It should be noted that in turbulent motion the solution of NS is strongly influenced by the initial conditions. It is valid for a specific case considered and does not provide any indication even for conditions close to those considered. The solution of a single specific case from the applicative point

of view can lose interest with respect to the evaluation of statistical quantities, such as averages and RMS. Therefore the best use of DNS techniques is in our opinion as a virtual laboratory. Indeed the DNS can be much more efficient than a physical laboratory, as it is able to determine a turbulent solution of the Navier Stokes equation, from which all relevant statistical quantities can be obtained. The only limit is the memory of the computer.

6.2. The Large Eddy Simulation (LES)

The main idea of the Large Eddy Simulation is that the length scales smaller than a given value are not considered in the numerical simulation. This omission is realized by filtering the motion equations, i.e. by considering the product of the motion equations times a filter function and integrating them on the whole fluid domain. By filtering the motion equations, the smallest length scales and relative eddy structures are removed and the governing equations for the filtered flow field are obtained. The smallest length scales are not ignored at all, but are determined in terms of the filtered, resolved field by means of suitable closure models, which account for the distribution of the turbulent kinetic energy among the spatial scales. The greatest removed length scale is the cut off scale, which is chosen so that most ($\sim 80\%$) of the turbulent kinetic energy is simulated. Of course, the LES of a given turbulent flow tends to the DNS of the same flow for smaller and smaller cut off length scales. Boundary and initial conditions must be assigned for the single case in exam.

6.2.1 The spatial filter operator

Let us consider the filtering operation with greater detail.
Let $G = G(\mathbf{r}, \mathbf{x})$ be a function of the point $\mathbf{x}$ and of the vector $\mathbf{r}$ $(\mathbf{r} = \mathbf{x} - \mathbf{x_0})$ distance between the points $\mathbf{x}, \mathbf{x_0}$. G is the filter or kernel function and satisfies the fundamental property:

$$\int_{\mathcal{D}} G(\mathbf{r},\mathbf{x})\, d^3\mathbf{r} = 1 \tag{6.4}$$

Where $\mathcal{D}$ is the whole fluid domain. Let $f(\mathbf{x},t)$ a flow quantity whatever. The filtered quantity is defined as:

$$\{f\} = \int_{\mathcal{D}} f(\mathbf{x}-\mathbf{r},t)G(\mathbf{r},\mathbf{x})\, d^3\mathbf{r} \tag{6.5}$$

The kernel function is said to be homogeneous if it does not depend on $\mathbf{x}$ and isotropic if it only depends on the vector modulus $r = |\mathbf{x}-\mathbf{x}_0|$ and not on the direction of the distance vector $\mathbf{r}$. In the following it will be assumed that the kernel function is both homogeneous and isotropic: $G = G(r)$. In order to highlight the filtering operation defined by equation 6.5, let us consider the singular kernel Dirac δ in 1D, for the sake of simplicity, defined as:

$$\delta(r) = 0 \; for \;\; r \neq 0; \;\; \delta(r) \to \infty \; for \;\; r \to 0; \; \int_0^\infty \delta(r)dr = 1 \tag{6.6}$$

The Dirac δ is a singular function, vanishing everywhere except at $x = x_0$, where it tends to infinity, whose definite integral on the whole real axis is finite and equal to 1. The most important property of the Dirac δ is that the filtered function $\{f\}$ coincides with the function itself:

$$\{f(x,t)\} = \int_0^\infty f(x-r,t)\, \delta(r)\, dr = f(x,t) \tag{6.7}$$

A more regularized version of the Dirac δ is the rectangular window (figure 6.1):

$$G(r) = 0 \;\; for \;\; |r| > \frac{\Delta}{2}; \;\; G(r) = \frac{1}{\Delta} \; for \;\; |r| \leq \frac{\Delta}{2} \tag{6.8}$$

which can be applied on finite domains. Δ is the basis of the rectangle.

The rectangular window tends to the Dirac δ function for vanishing Δ.
The filtered function $\{f\}$ coincides with the average value of the function over the interval Δ (figure 6.1).
Kernel functions may have different shapes, e.g. triangular, gaussian, etc., which affect the way with which the function is filtered. A kernel function which removes exactly the contributions due to the wavenumbers k greater than the cut off wavenumber k_c, leaving unaltered the contributions of the wavenumbers $k < k_c$, is the sharp cut off kernel:

$$G(r) = \frac{1}{\pi r} \sin\frac{\pi r}{\Delta} \tag{6.9}$$

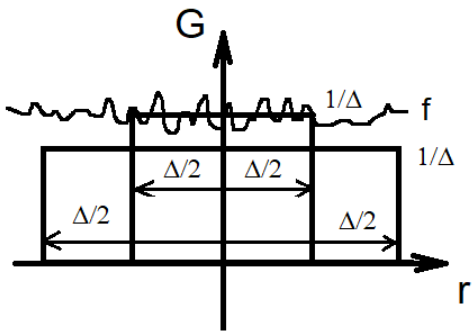

Figure 5.1: The rectangular window kernel.

Which eliminates exactly all the wavenumbers k greater than the cut off wavenumber $k_c = \pi/\Delta$.

6.2.2 Properties of the filtered and residual variable

The difference between the quantity f and the filtered quantity $\{f\}$ is defined as the residual f':

$$f' = f - \{f\} \tag{6.10}$$

by applying the filtering operation to the residual f', the filtered residual $\{f'\}$ is obtained:

$$\{f'\} = \{f\} - \{\{f\}\} \tag{6.11}$$

The latter does not vanish generally[9], i.e. the application of the filter to the filtered function $\{f\}$ does not give back the filtered function. In other words, the filtering operation does not act as the ensemble average operation.
Furthermore, the ensemble average operator commutes, i.e. it allows the inversion of sequence of operations with respect to spatial or temporal differential operators.

6.2.3 The filtered equations

Similarly to the Reynolds procedure, the filtering operation is applied to the equations of motion, so that the filtered equations for the new filtered dependent variables are obtained.
Let us consider the mass conservation equation for an incompressible turbulent flow and let us filter both sides:

$$\{\boldsymbol{\nabla} \cdot \mathbf{u}\} = \mathbf{0} \tag{6.12}$$

For the sake of simplicity, the filter is considered homogeneous and isotropic. The spatial differentiation can be exchanged with the filtering operation:

$$\boldsymbol{\nabla} \cdot \{\mathbf{u}\} = 0 \tag{6.13}$$

Equation 6.13 is the mass conservation equation for the incompressible filtered velocity fields. Expressing the velocity field as the sum of the filtered and residual velocity fields, the mass conservation equation for an incompressible turbulent flow takes the form:

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \boldsymbol{\nabla} \cdot (\{\mathbf{u}\} + \mathbf{u}') = \boldsymbol{\nabla} \cdot \{\mathbf{u}\} + \boldsymbol{\nabla} \cdot \mathbf{u}' = 0 \tag{6.14}$$

[9] Obviously, the residual variable of the sharp cut-off filter is zero.

Accounting for equation 6.13, it follows that the residual velocity field is also divergence free.
Let us apply the filtering operation to the Navier-Stokes equation term by term:

$$\frac{\partial\{\mathbf{u}\}}{\partial t} + \boldsymbol{\nabla}\cdot\{\mathbf{u}\otimes\mathbf{u}\} = -\frac{1}{\rho}\boldsymbol{\nabla}\{p\} + \{\mathbf{f}_b\} + \frac{\mu}{\rho}\boldsymbol{\nabla}^{\mathbf{2}}\{\mathbf{u}\} \tag{6.15}$$

The term $\{\mathbf{u}\otimes\mathbf{u}\}$ is usually expressed as:

$$\{\mathbf{u}\otimes\mathbf{u}\} = \{\mathbf{u}\}\otimes\{\mathbf{u}\} + \frac{1}{\rho}\mathbb{T}_{sg} \tag{6.16}$$

being $\mathbb{T}_{sg}$ the second order subgrid stress tensor, which accounts for the difference between $\{\mathbf{u}\otimes\mathbf{u}\}$ and $\{\mathbf{u}\}\otimes\{\mathbf{u}\}$. Taking into account 6.16, the filtered Navier-Stokes equation can be rewritten as:

$$\frac{\partial\{\mathbf{u}\}}{\partial t} + \boldsymbol{\nabla}\cdot(\{\mathbf{u}\}\otimes\{\mathbf{u}\}) = -\frac{1}{\rho}\boldsymbol{\nabla}\{p\} + \{\mathbf{f}_b\} + \frac{\mu}{\rho}\boldsymbol{\nabla}^{\mathbf{2}}\{\mathbf{u}\} - \frac{1}{\rho}\boldsymbol{\nabla}\cdot\mathbb{T}_{sg} \tag{6.17}$$

The second order subgrid stress tensor has to be expressed in function of the filtered variables. Joseph Smagorinsky, an American meteorologist, proposed the following closure equation:

$$\mathbb{T}_{sg} = \mathbb{T}_{sg}^{R} + \frac{1}{3}Tr\left(\mathbb{T}_{sg}\right)\mathbb{I}, \qquad \mathbb{T}_{sg}^{R} = -2\rho\nu_T\{\mathbb{D}\} \tag{6.18}$$

being $\mathbb{T}_{sg}^{R}$ the residual second order subgrid stress tensor and $Tr\left(\mathbb{T}_{sg}\right)$ the sum of the main diagonal elements of $\mathbb{T}_{sg}^{R}$. $\mathbb{I}$ is the identity second order tensor and $\{\mathbb{D}\}$ symmetric part of the deformation velocity second order tensor, defined in terms of the filtered velocity field. Finally, ν_T is the turbulent subgrid viscosity. Assuming the Smagorinsky closure equation 6.18, the filtered Navier-Stokes equation can be rewritten as:

$$\frac{\partial\{\mathbf{u}\}}{\partial t} + \boldsymbol{\nabla}\cdot(\{\mathbf{u}\}\otimes\{\mathbf{u}\}) = -\frac{1}{\rho}\boldsymbol{\nabla}\tilde{p} + \{\mathbf{f}_b\} + \frac{\mu}{\rho}\boldsymbol{\nabla}^2\{\mathbf{u}\} + 2\nu_T\boldsymbol{\nabla}\cdot\{\mathbb{D}\} \tag{6.19}$$

The pressure $\tilde{p}$ is defined as: $\tilde{p} = \{p\} + Tr(\mathbb{T}_{sg})/3$.
According to Smagorinsky, the turbulent subgrid viscosity can be defined as:

$$\nu_T = C_S\Delta^2\sqrt{\{\mathbb{D}\}:\{\mathbb{D}\}} \tag{6.20}$$

Where Δ is the width of the rectangular window filter (as shown e.g. in figure 6.1) and represents the order of magnitude of the largest non resolved lengthscale. C_S is the Smagorinsky constant, which for homogeneous and isotropic turbulence assumes the value $C_S = 0.18$.

6.3. Reynolds average Navier-Stokes based models

The Reynolds average Navier-Stokes (RANS) equation 4.9 is the starting point for many turbulence models. The RANS models represent turbulent hydrodynamic fields in terms of ensemble averaged variables. One of the advantages of the RANS models, having average quantities as unknowns, consists in the possibility of imposing conditions that are valid only for the average quantities, such as for example stationarity and symmetry.
The problem is solved numerically once a suitable closure model is adopted for the Reynolds stress tensor $\langle\mathbf{u}'\otimes\mathbf{u}'\rangle$. The latter can be expressed in terms of the ensemble average velocity field by means of the Boussinesq's hypothesis:

$$\langle\mathbf{u}'\otimes\mathbf{u}'\rangle = \frac{2}{3}\langle K'\rangle\mathbb{I} + \langle\mathbf{u}'\otimes\mathbf{u}'\rangle^R, \qquad \langle\mathbf{u}'\otimes\mathbf{u}'\rangle^R = -2\nu_T\langle\mathbb{D}\rangle \tag{6.21}$$

Where $\langle K'\rangle = <\mathbf{u}'\cdot\mathbf{u}'>/3$ is the ensemble average of the turbulent kinetic energy, $\langle\mathbf{u}'\otimes\mathbf{u}'\rangle^R$ the residual Reynolds stress tensor and ν_T is the turbulent viscosity. Thanks to the Boussinesq's decomposition 6.21 the

RANS equation 4.9 can be rewritten as:

$$\frac{\partial \langle \mathbf{u} \rangle}{\partial t} + \langle \mathbf{u} \rangle \cdot \boldsymbol{\nabla} \langle \mathbf{u} \rangle = -\frac{\boldsymbol{\nabla} \bar{p}}{\rho} + \mathbf{f}_b + \frac{\mu}{\rho} \boldsymbol{\nabla}^2 \langle \mathbf{u} \rangle + 2\boldsymbol{\nabla} \cdot (\nu_T \langle \mathbb{D} \rangle) \tag{6.22}$$

Where the pressure $\bar{p}$ is defined as: $\bar{p} = \langle p \rangle + 2\langle K' \rangle / 3$.
The core of the Boussinesq's hypothesis is the definition of the turbulent viscosity ν_T. The complexity and the reliability of the RANSE-based model depends on the additional equations used to obtain the turbulent viscosity.

6.3.1 Algebraic models

The turbulent viscosity is defined by the algebraic equation:

$$\nu_T = k_a u_0 \delta \tag{6.23}$$

Where u_0, δ and k_a are respectively the characteristic length, the velocity scale and a constant, characteristic for the considered flow.
Algebraic models are the simplest and work only for simple flows, mainly for uniform turbulent flows for which a spatial dimension is prevailing on the others, as open channel flows, jets, etc. These simple models are widely used e.g. in modelling river flows.
A very famous algebraic model, which permitted to obtain good results in turbulent pipe and open channel flows, is the mixing length model. The latter, due to Prandtl, is generally adopted for a shear flow characterized by the ensemble average velocity component $\langle u \rangle$ depending on the coordinate y defined along the direction perpendicular to the velocity:

$$\nu_T = \kappa^2 \ell^2 \left| \frac{\partial \langle u \rangle}{\partial y} \right| \tag{6.24}$$

The modulus is applied to ensure that ν_T is positive. κ is the Von-Karman constant, while ℓ is the mixing length, a length scale representative for the fluctuating flow field. Considering e.g. the turbulent uniform flow on a flat

plate, the mixing length ℓ can be assumed to be proportional to the distance y from the flat plate along the perpendicular direction to it. One of the most important results obtained applying the mixing length hypothesis 6.24 is the logarithmic velocity profile in uniform turbulent pipes and open channel flows.

6.3.2 The $K-\epsilon$ model

The algebraic model gives a very simplified description of the turbulent viscosity, acceptable only for uniform turbulent shear flows. As soon as the turbulent flow becomes more complicated, for example, due to the complexity of the flow domain geometry, a correspondingly more complex turbulent model has to be used. A very popular one is the $K-\epsilon$ model, where, for the sake of simplicity, K indicates the ensemble average turbulent kinetic energy: $K=\langle K' \rangle$, with dimension L^2T^{-2}, and ϵ the dissipated turbulent kinetic energy per unit mass and time: $\epsilon = 2\nu\langle \mathbb{D}' : \mathbb{D}' \rangle$, with dimension L^2T^{-3}.

The turbulent viscosity can be modelled after Prandtl (1945)[10] and followers as:

$$\nu_T = k_\nu \frac{K^2}{\epsilon} \tag{6.25}$$

Where k_ν is a constant. In turn K, ϵ can be determined by solving suitable equations.

As for the determination of K, the balance of the ensemble average turbulent kinetic energy average turbulent kinetic energy 5.17:

[10] Prandtl proposed to consider the turbulent viscosity ν_T proportional to $L^{-1}K^{1/2}$, where L is a length characterizing the large-scale turbulent motion. Furthermore, for high Reynolds, the viscous dissipation tends to a limit independent of the viscosity itself but linked to K and L. So dimensionally ϵ can be expressed by $L^{-1}K^{3/2}$ and from which, by eliminating L, equation 6.25 is obtained.

$$\frac{\partial\langle K'\rangle}{\partial t}+\langle\mathbf{u}\cdot\boldsymbol{\nabla}K'\rangle=-\frac{1}{\rho}\boldsymbol{\nabla}\cdot(\langle p'\mathbf{u}'\rangle-2\mu\langle\mathbb{D}'\mathbf{u}'\rangle)-\langle\mathbf{u}'\otimes\mathbf{u}'\rangle:\boldsymbol{\nabla}\langle\mathbf{u}\rangle-\frac{2\mu}{\rho}\langle\mathbb{D}':\mathbb{D}'\rangle$$

is adopted with the following modifications.

With reference to equation 5.17, the convective term at left hand side is expressed as:

$$\langle\mathbf{u}\cdot\boldsymbol{\nabla}K'\rangle=\boldsymbol{\nabla}\cdot(\langle\mathbf{u}\rangle K)+\boldsymbol{\nabla}\cdot\langle\mathbf{u}'K'\rangle \tag{6.26}$$

The last term at right hand side can be inserted in the first term at right hand side of equation 5.17.
Moreover, introducing the diffusive term:

$$\rho\frac{\nu_T}{\sigma_K}\boldsymbol{\nabla}K=-\langle p'\mathbf{u}'\rangle+2\mu\langle\mathbb{D}'\mathbf{u}'\rangle-\langle\mathbf{u}'K'\rangle \tag{6.27}$$

Equation 5.17 assumes the form:

$$\frac{\partial K}{\partial t}+\boldsymbol{\nabla}\cdot(\langle\mathbf{u}\rangle K)=\boldsymbol{\nabla}\cdot\left(\frac{\nu_T}{\sigma_K}\boldsymbol{\nabla}K\right)-\langle\mathbf{u}'\otimes\mathbf{u}'\rangle:\boldsymbol{\nabla}\langle\mathbf{u}\rangle-\frac{2\mu}{\rho}\langle\mathbb{D}':\mathbb{D}'\rangle \tag{6.28}$$

Where σ_K is a dimensionless constant.
The production term $-\langle\mathbf{u}'\otimes\mathbf{u}'\rangle:\boldsymbol{\nabla}\langle\mathbf{u}\rangle$, using the Boussinesq's decomposition 6.21, can be expressed as:

$$-\langle\mathbf{u}'\otimes\mathbf{u}'\rangle:\boldsymbol{\nabla}\langle\mathbf{u}\rangle=2\,\nu_T\langle\mathbb{D}\rangle:\langle\mathbb{D}\rangle\equiv P \tag{6.29}$$

Where P is known as production of turbulent kinetic energy. Accounting for previous consideration, the modified balance equation for the ensemble average turbulent kinetic energy $\langle K'\rangle$ can be rewritten as:

$$\frac{\partial K}{\partial t}+\boldsymbol{\nabla}\cdot(\langle\mathbf{u}\rangle K)=\boldsymbol{\nabla}\cdot\left(\frac{\nu_T}{\sigma_K}\boldsymbol{\nabla}K\right)+2\,\nu_T\langle\mathbb{D}\rangle:\langle\mathbb{D}\rangle-\epsilon \tag{6.30}$$

As for the determination of ϵ the exact equation for this quantity could be obtained after a cumbersome and tedious calculation with additional unknown variables. Hence, it is preferable to use an empirical equation, defined as follows:

$$\frac{\partial \epsilon}{\partial t} + \boldsymbol{\nabla} \cdot (\langle \mathbf{u} \rangle \epsilon) = \boldsymbol{\nabla} \cdot \left(\frac{\nu_T}{\sigma_\varepsilon} \boldsymbol{\nabla} \epsilon \right) + 2\, c_{1\varepsilon}\, \nu_T \frac{\langle \mathbb{D} \rangle : \langle \mathbb{D} \rangle}{K} - c_{2\varepsilon} \frac{\epsilon^2}{K} \qquad (6.31)$$

The $K - \epsilon$ model consists of equations 6.25, 6.30 and 6.31. They depend on the five empirical dimensionless constants $k_\nu, \sigma_K, \sigma_\varepsilon, c_{1\varepsilon}, c_{2\varepsilon}$, which must be chosen carefully in order to reproduce a given turbulent flow.
This is exactly the limitation of this model: it needs an accurate calibration, which is valid for the considered case, i.e. it does not have a universal validity.

6.3.3 The Reynolds stress model

In the RANS equations, the six components of the Reynolds stress tensor represent additional unknowns, which can be determined by using six additional equations. Such a turbulence model, not based on the Boussinesq's hypothesis, is the Reynolds stress model. The additional equations for the stress components can be obtained by means of a rather cumbersome procedure, which will be illustrated in the following. Let us subtract the RANS equation from the Navier-Stokes equation term by term, both written in component form. The equation for the residual velocity field u_i' is obtained:

$$\frac{\partial u_i'}{\partial t} + \frac{\partial}{\partial x_k}(u_i' \langle u_k \rangle + \langle u_i \rangle u_k' + u_i' u_k') = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_k \partial x_k} + \frac{\partial \langle u_i' u}{\partial x_k} \qquad (6.32)$$

Let us multiply equation 6.32 times u_j':

$$u_j' \frac{\partial u_i'}{\partial t} + u_j' \frac{\partial}{\partial x_k}(u_i' \langle u_k \rangle + \langle u_i \rangle u_k' + u_i' u_k') = -\frac{u_j'}{\rho}\frac{\partial p'}{\partial x_i} + \nu u_j' \frac{\partial^2 u_i'}{\partial x_k \partial x_k} + u_j' \frac{\partial \langle u_i' u_k' \rangle}{\partial x_k} \tag{6.33}$$

Then equation 5.33 is rewritten changing index i with j and j with i:

$$u_i' \frac{\partial u_j'}{\partial t} + u_i' \frac{\partial}{\partial x_k}\left(u_j' \langle u_k \rangle + \langle u_j \rangle u_k' + u_j' u_k'\right) = -\frac{u_i'}{\rho}\frac{\partial p'}{\partial x_j} + \nu u_i' \frac{\partial^2 u_j'}{\partial x_k \partial x_k} + u_i' \frac{\partial \langle u_j' u_k' \rangle}{\partial x_k} \tag{6.34}$$

Summing and ensemble averaging equations 6.33 and 6.34, the following equation is obtained for the Reynolds stress $\langle u_i' u_j' \rangle$:

$$\frac{\partial \langle u_i' u_j' \rangle}{\partial t} + \langle u_k \rangle \frac{\partial \langle u_i' u_j' \rangle}{\partial x_k} + \langle u_i' u_k' \rangle \frac{\partial \langle u_j \rangle}{\partial x_k} + \langle u_j' u_k' \rangle \frac{\partial \langle u_i \rangle}{\partial x_k} + \frac{\partial \langle u_i' u_j' u_k' \rangle}{\partial x_k} = \nu \frac{\partial^2 \langle u_i' u_j' \rangle}{\partial x_k \partial x_k} - \langle \frac{u_i'}{\rho}\frac{\partial p'}{\partial x_j} + \frac{u_j'}{\rho}\frac{\partial p'}{\partial x_i} \rangle - 2\nu \langle \frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k} \rangle \tag{6.35}$$

The Reynolds stress equation 6.35 is the most complete classical turbulence model. Indeed, the eddy-viscosity hypothesis is avoided and every single component of the Reynolds stress tensor is directly computed.

The Reynolds stress model uses the exact stress transport equation and offers better accuracy than eddy-viscosity based turbulence models at a lower computational price than Direct Numerical and Large Eddy Simulations.

Nonetheless, equation 6.35 is very complicated. The most complex aspect is the need of suitable closure equations for the third order correlation term at left hand side, the pressure-velocity correlation and the dissipation term at right hand side.

CHAPTER 7 RECENT PERSPECTIVES IN THE TURBULENCE STUDY AND THE DETERMINISTIC CHAOS

7.1. Prologue

In 1963, the meteorologist Edward N. Lorenz of the Massachusetts Institute of Technology discovered that the following nonlinear system of first order ordinary differential equations:

$$\begin{aligned} \frac{dx_1}{dt} &= a(x_2 - x_1) \\ \frac{dx_2}{dt} &= bx_1 - x_2 - x_1x_3 \\ \frac{dx_3}{dt} &= x_1x_2 - cx_3 \end{aligned} \tag{7.1}$$

where a, b, c are given parameters, has solutions with apparently irregular, disordered, and unforeseeable behaviour, although it is a deterministic system with no random variables. Lorenz was a meteorologist who knew mathematics and had considered this system in order to simulate and predict the actual functioning of atmospheric phenomena on a simplified level.

Lorenz observed that, starting with slightly different initial conditions, were obtained completely different solutions. The solutions, even defined trajectories in the three-dimensional space $Ox_1x_2x_3$, draw an object now named the Lorenz's butterfly (figure 7.1) in honour of Lorenz. It has a well-defined shape with two wings. The state of the system at the generic instant, that is the set of values assumed by the solution at that generic instant, oscillates over time from one wing to the other without any rule. It moves in an apparently random way, erratic, unpredictable.

This discovery seemed to contradict the statement attributed to Laplace, which until then had never been never questioned, that *given the precise knowledge of the initial conditions, it should be possible to predict the future of the universe.*

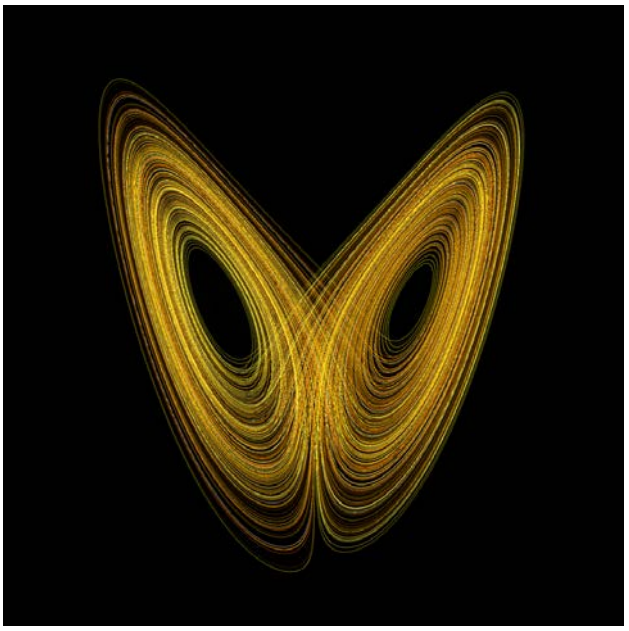

Figure 7.1: The Lorenz's butterfly.
Source: https://commons.wikimedia.org/wiki/File:Lorenz_system_r28_s10_b2-6666.png

In other words by assigning the initial conditions it is possible to foresee the future behaviour of a phenomenon for which the governing deterministic equations are known. The problem is with what precision the initial conditions have to be known and the calculations have to be made. In this sense, Laplace's statement is still valid.

Lorenz's discovery put into discussion the scenario of Landau (1908-1968) and Hopf (1902-1983) relative to the mechanism of the transition to turbulence. According to Landau and Hopf, a weakly turbulent fluid, i.e. at the beginning of the transition, is described by the superposition of a small number of modes, the excited or first unstable ones, showing no sensitivity to the initial conditions. The number of excited modes increases by successive bifurcations from the already excited modes to new ones with increasing Reynolds number. The excited modes are more and more distributed over the whole spectrum as the Reynolds number increases. Some of them have their periodicity in irrational ratio and their overlapping

generates non-periodic but quasi-periodic trends, thus justifying the non-periodicity of the phenomenon. In 1971 the two scientists David Ruelle and Floris Takens hypothesized a turbulent flow mechanism similar to that found by Lorentz with his simple dynamical system in opposition to the well-consolidated Landau-Hopf theory of the continuous excitation of Fourier's modes. In 1979, Lorentz at a meteorological meeting in Washington presented his paper *Can the beating of butterfly wings in Brazil cause a tornado in Texas?* These words have had a huge resonance, making the theory of deterministic chaos very popular. Lorenz realized the connection between aperiodicity and unpredictability and convinced himself that long-term weather forecasts were impossible. Initial variables cannot be known exactly, and in any case, even with the help of powerful computers, the unavoidable approximated calculation would have caused enormous changes in the results. Today, despite the great development of the computational resources, we know that Lorenz was right; weather forecasts more advanced than a week are unreliable.

Let's go back to Ruelle and Takens. They introduced the concept of strange attractor for objects such as the Lorenz's butterfly. These objects are attractors in the sense that they have a basin of attraction: i.e. if the initial conditions belong to the basin, the solution is attracted to it and after a while it goes to the attractor and remains there indefinitely, seemingly wandering without rules. These attractors are, however, special: this is why they are defined strange. They have not an integer geometric dimension: indeed they are neither points, nor lines, nor surfaces, nor volumes. They are something in between, that is, they have a non-integer, fractal geometric dimension. Two trajectories, even originating from two almost identical initial conditions, separate significantly after a while, moving away from each other exponentially with time.

Lorenz was able to observe the strong dependence on initial conditions with his model, i.e. the chaotic behaviour. Starting from different initial conditions one gets completely different trajectories in the attractor and, if one does not know the initial conditions, cannot say at a given instant of

time which is the state of the system, i.e. which is the position of the trajectory on the attractor.
Research in this field since then on has continued. Considering the equations governing a particular phenomenon, one tries to characterize the attractor, or the attractors, to evaluate whether or not it is a fractal object, i.e. wheter it is strange or not and whether the state of the system belongs to certain zones of the attractor more than others.
In this way, one can estimate the probability that the state of the system stays in a given zone of the attractor rather than in another. This latter aspect is a positive and constructive approach, in order to overcome the problem of the strong dependence on the initial conditions.
In what follows, we will try to simplify and present a few arguments and concepts used in the deterministic chaos theory.

7.2. Differential equations systems

7.2.1 Hydrodynamic Analogy

It is worth recalling the fundamentals of the theory of linear differential equations systems. Let consider a system of linear differential equations with real constant coefficients a_{ij}:

$$\frac{dx_i}{dt} = a_{ij}x_j \qquad i = 1, \dots, m, \qquad j = 1, \dots, m \tag{7.2}$$

In vector form it can be expressed as:

$$\frac{d\mathbf{x}}{dt} = \mathbb{A}\,\mathbf{x} \tag{7.3}$$

Where $\mathbb{A}$ is the 2nd order tensor with scalar components a_{ij} and $\mathbf{x}$ is the vector representing a position in the m-dimensional space:

$$\mathbf{x} = \{x_1, x_2, \dots, x_m\} \tag{7.4}$$

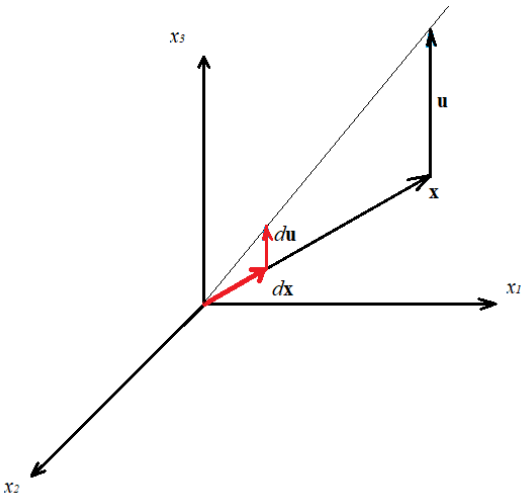

Figure 7.2: Vectors $\mathbf{x}, \mathbf{u}, d\mathbf{x}, d\mathbf{u}$ in three-dimensional space.

$d\mathbf{x}/dt$ represents the velocity $\mathbf{u}$ with which the position defined by $\mathbf{x}$ moves in the *m*-dimensional space as time goes by. A hydrodynamic analogy with the motion of a fluid body can be introduced as follows. The vector $\mathbf{u}$ can represent the velocity of the fluid particle:

$$u_i = a_{ij} x_j \tag{7.5}$$

Then the deformation velocity can be expressed as:

$$du_i = a_{ij} dx_j \tag{7.6}$$

or in vector form:

$$d\mathbf{u} = \mathbb{A}\, d\mathbf{x} \tag{7.7}$$

The tensor $\mathbb{A}$ can be expressed as the sum of the symmetric tensor $\mathbb{D}$ and the antisymmetric tensor $\mathbb{W}$:

$$d\mathbf{u} = (\mathbb{D} + \mathbb{W})d\mathbf{x} = \mathbb{D}\, d\mathbf{x} + \mathbb{W}\, d\mathbf{x} \tag{7.8}$$

The fluid flow expressed by equation 7.2 is a peculiar one. In fact, thanks to the fact that the coefficients a_{ij} are constant with respect to space and time, the flow structure does not depend on the spatial scale. This observation holds for the deformation velocity 7.8 too.

7.2.2 Eigenvalues and Eigenvectors

According to equation 7.5, the vector $\mathbf{u}$ can be seen as the linear transformation of the vector $\mathbf{x}$. In general vector $\mathbf{u}$ is not parallel to vector $\mathbf{x}$. If $\mathbf{u} = \mathbf{u}^*$ and $\mathbf{x} = \mathbf{x}^*$ are parallel, the following condition is satisfied:

$$\mathbf{u}^* = \mathbb{A}\,\mathbf{x}^* = \lambda\,\mathbf{x}^* \tag{7.9}$$

Where λ is a scalar constant and it is defined as the eigenvalue of the 2nd order tensor $\mathbb{A}$.

The vector $\mathbf{x}^*$ is defined eigenvector of the 2nd order tensor $\mathbb{A}$. Eigenvalues can be real or complex. In case of a complex eigenvalue, the corresponding eigenvector is also complex. As $\mathbb{A}$ is real, for each complex eigenvalue and eigenvector, the corresponding conjugated eigenvalue and eigenvector exist.

7.2.3 Exponential function of a matrix

Given an $n \times n$ square matrix $\mathbb{A}$ and a function $f = f(x)$, that can be developed in power series:

$$f(x) = \sum_{i=0}^{\infty} c_i\, x^i \tag{7.10}$$

the matrix function $f = f(\mathbb{A})$ is defined as follows:

$$f(\mathbb{A}) = \sum_{i=0}^{\infty} c_i \, \mathbb{A}^i \tag{7.11}$$

Let us consider the function $f = e^x$, whose power series expansion is given by:

$$e^x = 1 + \sum_{i=0}^{\infty} \frac{x^i}{i!} \tag{7.12}$$

The corresponding matrix function $e^{\mathbb{A}}$ is then given by:

$$e^{\mathbb{A}} = \mathbb{I} + \sum_{i=0}^{\infty} \frac{\mathbb{A}^i}{i!} \tag{7.13}$$

Where $\mathbb{I}$ is the unity matrix.
If $\mathbb{A}$ is diagonal (e.g. the matrix derives from a diagonalization):

$$\mathbb{A} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \tag{7.14}$$

where $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of the matrix $\mathbb{A}$.
Then:

$$e^{\mathbb{A}} = \begin{bmatrix} e^{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n} \end{bmatrix} \tag{7.15}$$

and

$$e^{\mathbb{A}t} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \tag{7.16}$$

The time derivative of the matrix function $e^{\mathbb{A}t}$ can be calculated as:

$$\frac{de^{\mathbb{A}t}}{dt} = \mathbb{A}\, e^{\mathbb{A}t} \tag{7.17}$$

The previous result can be used to express the solution of a linear ordinary differential equations system with constant coefficients:

$$\mathbf{x} = \mathbf{x}_0\, e^{\mathbb{A}t} \tag{7.18}$$

where $\mathbf{x}_0$ is the value of $\mathbf{x}$ at time $t = 0$. According to equation 7.18, the tensor $e^{\mathbb{A}t}$, that is the tensor obtained assuming the scalar components to be the elements of the matrix $e^{\mathbb{A}t}$, transforms the position vector at time $t = 0$, $\mathbf{x}_0$, in the position vector $\mathbf{x}$ at the generic time t. The Jacobian matrix $\partial\mathbf{x}/\partial\mathbf{x}_0$ coincides with $e^{\mathbb{A}t}$, does not depend on $\mathbf{x}_0$ and $\mathbf{x}$ and has the same eigenvalues. The Jacobian determinant is given by:

$$\left\|\frac{\partial\mathbf{x}}{\partial\mathbf{x}_0}\right\| = e^{\lambda_1 t} e^{\lambda_2 t} \dots e^{\lambda_n t} = e^{tr(\mathbb{A})t} \tag{7.19}$$

It is well known that the Jacobian determinant represents the ratio of the hyper volume at time t and the hypervolume at time t_0.
If $tr(\mathbb{A}) = 0$ the value of the hypervolume is conserved with respect to time and the system is defined as conservative, while if $tr(\mathbb{A}) < 0$ the value of the hypervolume decreases with time and the system is defined as dissipative.
In the hydrodynamic analogy between deterministic systems and fluid flow having a corresponding kinematics, the conservative system expresses or represents the incompressible fluid; the dissipative system (of volumes) represents a contracting compressible flow.

7.2.4 Definition of attractor

In a deterministic dissipative system, an attractor is an invariant and bounded subset of the space of the states to which the trajectories converge;

the trajectories start from another subset (called attraction basin), that contains the attractor. This means that the attractor must: a) possess a basin of attraction; b) not vary with time and cannot be divided into several attractors. From the second condition it follows that any area of the attractor will sooner or later be visited by the state of the system in its temporal evolution (otherwise the attractor would be divisible).

7.3. Lyapunov exponents

7.3.1 The first Lyapunov exponent

Let's consider the linear differential system:

$$\frac{d\mathbf{x}}{dt} = \mathbb{A}\,\mathbf{x} \tag{7.20}$$

Let us consider two trajectories in the space of states, whose distance at time $t = 0$ is $\delta\mathbf{x}_0$, being $\delta\mathbf{x}_0$ an infinitesimal vector corresponding to a small perturbation of the state $\mathbf{x}_0$. At generic time t the distance becomes $\delta\mathbf{x}(t)$.

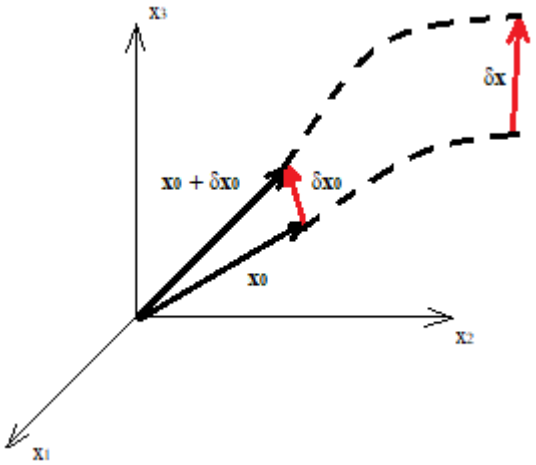

Figure 7.3: Two trajectories in the state space with distance $\delta\mathbf{x}(t)$.

The equations governing the evolution of $\delta\mathbf{x}(t)$ are:

$$\frac{d\delta\mathbf{x}(t)}{dt} = \mathbb{A}\,\delta\mathbf{x}(t) \tag{7.21}$$

and the solutions are:

$$\delta\mathbf{x}(t) = \delta\mathbf{x}_0\, e^{\mathbb{A}t} \tag{7.22}$$

Let's sort the eigenvalues of $\mathbb{A}$ in descending order: $\lambda_1, \lambda_2, \dots, \lambda_n$. In case of complex eigenvalues, the real part is considered. The modulus of $\delta\mathbf{x}(t)$ depends on its components $\delta x_k(t)$ but, as time increases, the contribution of $\delta x_1(t)$ prevails over the others:

$$\frac{\delta x_k(t)}{\delta x_1(t)} = \frac{\delta x_{0k}}{\delta x_{01}}\, e^{(\lambda_k-\lambda_1)t} \tag{7.23}$$

Indeed, the exponents $\lambda_k - \lambda_1$ are negatives and the exponential functions $e^{(\lambda_k-\lambda_1)t}$ decrease with t. Let us assume that $\delta\mathbf{x}_0$ is parallel to the direction of the eigenvector corresponding to the first eigenvalue. Then the modulus evolves as:

$$|\delta\mathbf{x}(t)| = \left|\delta\mathbf{x}_0\, e^{\mathbb{A}t}\right| = |\delta\mathbf{x}_0|\, e^{\lambda_1 t} \tag{7.24}$$

From which:

$$\frac{|\delta\mathbf{x}(t)|}{|\delta\mathbf{x}_0|} = e^{\lambda_1 t} \tag{7.25}$$

and

$$\lambda_1 = \frac{1}{t}\ln\frac{|\delta\mathbf{x}(t)|}{|\delta\mathbf{x}_0|} \tag{7.26}$$

The first eigenvalue λ_1 can be obtained by calculating the logarithm of the deformation at time t divided by t.

Let us divide the time t in N small intervals δt and let's consider the sequence:

$$t_1 = \delta t, t_2 = 2\delta t, \dots, t = N\delta t \tag{7.27}$$

Equation 7.25 becomes:

$$\frac{|\delta \mathbf{x}(t_1)|}{|\delta \mathbf{x}_0|} \frac{|\delta \mathbf{x}(t_2)|}{|\delta \mathbf{x}(t_1)|} \cdots \frac{|\delta \mathbf{x}(t)|}{|\delta \mathbf{x}(t_{N-1})|} = e^{\lambda_1 N \delta t} \tag{7.28}$$

From which

$$\lambda_1 = \frac{1}{N\delta t} \sum_{j=1}^{N} \ln \frac{|\delta \mathbf{x}(t_j)|}{|\delta \mathbf{x}(t_{j-1})|} = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\delta t} \ln \frac{|\delta \mathbf{x}(t_j)|}{|\delta \mathbf{x}(t_{j-1})|} = \frac{1}{N} \sum_{j=1}^{N} \lambda_{1j} \tag{7.29}$$

The λ_{1j} $(j = 1, \dots, N)$ are calculated for each t_j, perturbing the position reached by the trajectory $\mathbf{x}(t_j)$ at the instant t_j in the direction of the first eigenvector and evaluating the deformation ratio $|\delta \mathbf{x}(t_j)|/|\delta \mathbf{x}(t_{j-1})|$. In linear systems, the direction of the eigenvectors does neither change, nor does the deformation ratio. All the λ_{1j} are equal to λ_1.

The previous procedure can be used for a nonlinear deterministic system. The modules of the perturbative vectors are infinitesimal and, at each step of the procedure, it is possible to linearize the equations that regulate their temporal evolution. Nevertheless, in non-linear systems at each step the linearized system changes and, consequently, the direction of the eigenvectors and the deformation ratio change. Equation 7.29 should be considered in the limit of $N \to \infty$. The first **Lyapunov exponent** is thus given by:

$$\lambda_1 = \lim_{N \to \infty} \left(\frac{1}{N\delta t} \sum_{j=1}^{N} \ln \frac{|\delta \mathbf{x}(t_j)|}{|\delta \mathbf{x}(t_{j-1})|} \right) \tag{7.30}$$

7.3.2 The subsequent Lyapunov exponents

The number of Lyapunov exponents:

$$\lambda_1, \lambda_2, \dots, \lambda_m \tag{7.31}$$

is equal to the dimensions of the state space, i.e. the number of differential equations of the first order. The second Lyapunov exponent λ_2, can be obtained using the same procedure as the first, but erasing the component of $\delta\mathbf{x}_0$ along the first eigenvector:

$$\delta\mathbf{x}_0 = \{0, \delta x_{02}, \delta x_{03}, \dots\} \tag{7.32}$$

Thereby $\delta\mathbf{x}_0$ cannot stretch or shrink in the direction of the first eigenvector and the previous procedure leads to the determination of the second Lyapunov exponent. It is possible to continue in this way to obtain the subsequent exponents, but a better way can be followed. In fact, the evolution of an infinitesimal surface can be considered. The sum of the first two exponents can be obtained as:

$$\lambda_1 + \lambda_2 = \lim_{N\to\infty}\left(\frac{1}{N\delta t}\sum_{j=1}^{N}\ln\frac{\delta\sigma(t_j)}{\delta\sigma(t_{j-1})}\right) \tag{7.33}$$

With this aim, if a linear system and two infinitesimal vectors $\delta\mathbf{x}(t)$ and $\delta\mathbf{y}(t)$ parallel to the directions of the two first eigenvectors are considered, the area of the surface defined by the two vectors is proportional to the product of the modules. At time $t = 0$ is:

$$\delta\sigma_0 \approx |\delta\mathbf{x}_0||\delta\mathbf{y}_0| \tag{7.34}$$

At time t:

$$\mathbf{x}(t) = \delta\mathbf{x}_0 e^{\lambda_1 t}, \;\; \mathbf{y}(t) = \delta\mathbf{y}_0 e^{\lambda_2 t} \tag{7.35}$$

then

$$\delta\sigma(t) \approx |\delta\mathbf{x}_0||\delta\mathbf{y}_0|e^{(\lambda_1+\lambda_2)t} \tag{7.36}$$

Hence, because the constant of proportionality is always the same in time and space for linear systems:

$$\frac{\delta\sigma(t)}{\delta\sigma_0} = e^{(\lambda_1+\lambda_2)t} \tag{7.37}$$

Similarly, for $\lambda_1 + \lambda_2 + \lambda_3$ the deformation of an infinitesimal volume can be considered. It is possible to go on with this procedure, until the consideration of hypervolumes whose dimension is not larger than the dimension of the attractor.

The first Lyapunov exponent λ_1 represents the time average of the maximum growth rate of an infinitesimal perturbation of the state of the system on the attractor along its trajectory. The infinitesimal perturbation is an infinitesimal vector from a point of the attractor along its trajectory. The direction of this vector can change so that it is always directed in the direction of maximum growth. Hence, the first Lyapunov exponent λ_1 represents the average growth rate of the modulus of the infinitesimal vector. The sum of the first two Lyapunov exponents $\lambda_1 + \lambda_2$ represents the average maximum growth rate of the area of the parallelogram surface defined by two infinitesimal vectors as time goes on.

In general, the sum of the first k Lyapunov exponents $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ represents the growth rate of the k-dimensional hyper-volume defined by k infinitesimal linearly independent vectors as time goes by.

Hence, let us consider an m-dimensional sphere of radius δr around the state of the system. As time goes by the sphere changes shape and the measure of the hyper-volume occupied on the m-dimensional space. At each time of the temporal sequence, already considered, $t_1 + t_2 + \cdots + t_N$ the new hypervolume can be obtained by the Jacobian transform. At time t_1 the sphere becomes an ellipsoid with semi-axis coinciding with the principal directions. The semi-axis are stretched or shortened by $e^{\lambda_1 t}$, $e^{\lambda_2 t}$, …, $e^{\lambda_m t}$. If

the system is linear, as time goes by the ellipsoid deforms keeping its axis aligned with the same directions. The length of the axis increases if the eigenvalue is positive and decreases if the eigenvalue is negative. If all the eigenvalues are negative, the ellipsoid becomes a point. If the first eigenvalue is zero and the others negative, the length of the corresponding semi-axis is constant with time, while the other lengths decrease. In other words, the ellipsoid collapses onto a segment. If the first two eigenvalues are zero and the others negative, the lengths of the corresponding semi-axis are constant with time, while the others reduce with time more and more. In this case, the ellipsoid collapses onto a flat surface. The spatial dimensions of the set onto which the ellipsoid collapses increase with the number of vanishing eigenvalues.

If the system is nonlinear, at each instant of time eigenvalues and eigenvectors change. The first Lyapunov exponent is the limit to which the average of the first eigenvalues of the procedure of the temporal sequence tends. A similar procedure determines the other Lyapunov exponents. A single transform from initial time t_1 to the generic time t can be considered. The limit of the average operation determines the unique linear transform, whose eigenvalues are: $e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_m t}$ where $\lambda_1, \dots, \lambda_m$ are the Lyapunov exponents.

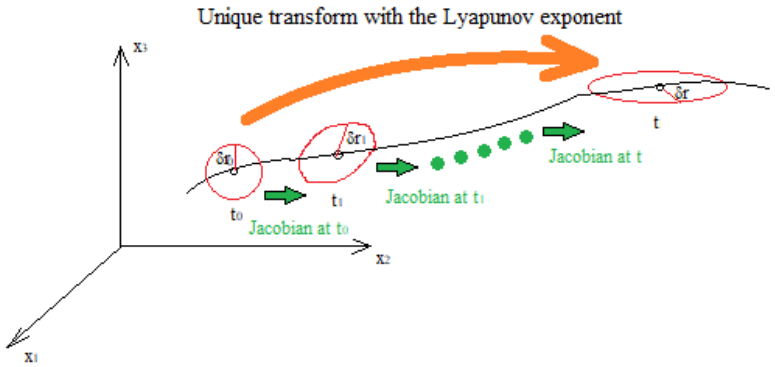

Figure 7.4: m -dimensional sphere in the m-dimensional space changing shape and measures of the hyper-volume as time increases.

7.4. Geometric properties of attractors

7.4.1 Lyapunov exponents and attractors

The Lyapunov exponents characterize the attractor. If the Lyapunov exponents are all negative, the attractor has dimension 0 and is called fixed point. If the first Lyapunov exponent is zero and the others are negative, the attractor has dimension 1 and is called limit cycle (fig. 7.5).

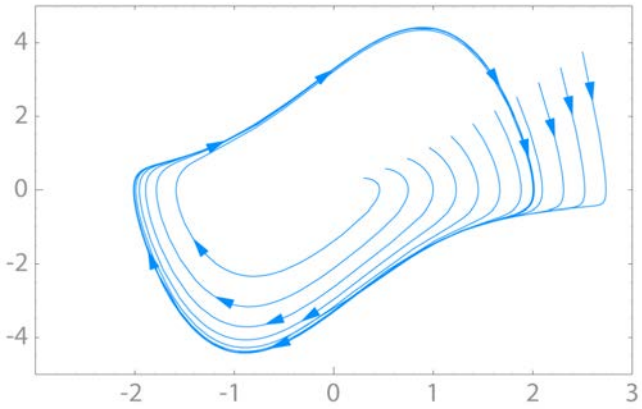

Figure 7.5: Limit cycle.

Source: https://it.wikipedia.org/wiki/Attrattore#/media/File:VanDerPolPhaseSpace.png

If the first two Lyapunov exponents are zero and the others are negative, the attractor has dimension 2 and is called two-dimensional torus (fig. 7.6).

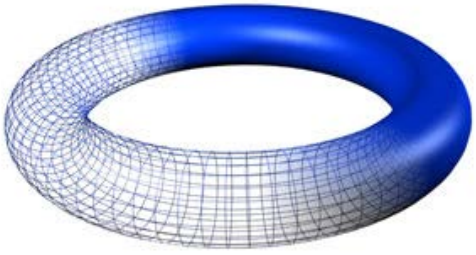

Figure 7.6: 2D Torus.

Source: https://it.wikipedia.org/wiki/Attrattore#/media/File:Torus.jpg

If the first k Lyapunov exponents are zero and the others are negative, the attractor has dimension k and is called k-dimensional torus. Note that the dimensions quoted are all integer numbers. Since the deterministic system of departure is dissipative, a set of initial conditions forming a starting hyper volume of dimensions m in the basin of attraction is reduced by the transforms on time to a set of dimensions n ($n < m$) of the attractor. A starting hyper volume on the attractor having the same dimensions n of the attractor, is transformed in a new set which maintains the dimensions n as time goes by. In this sense, the attractor is conservative, with the first n exponents of Lyapunov equal to zero.

A sphere of infinitesimal radius of initial conditions, contained in the attractor, evolves over time deforming, lengthening, then shortening, and then lengthening again, without ever getting too far from the spherical shape so that all the Lyapunov exponents are vanishing. Thus, the influence of the initial conditions is weak. For example, if the motion is periodic, limit cycle, or almost periodic, torus, the points of the sphere can move away but then they have to get close again as shown in the following figure (fig. 7.7).

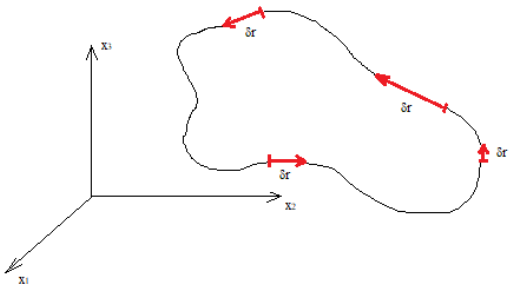

Figure 7.7. Influence of initial conditions on the limit cycle.

7.4.2 Strange attractors

It may happen that there are positive Lyapunov exponents. In this case a sphere is deformed with time, becoming ever longer and thinner, similar to a *spaghetto* or a thread, but always remaining in the attractor. This means that the initial conditions have a strong influence. As the attractor is bounded,

the *spaghetto* curls on itself; the state of the system cannot achieve the same values achieved in the past, otherwise the motion would be periodic and the trajectory would be a single closed filament. Therefore, the *spaghetto* will continue indefinitely to curl on itself (folding) and the state of the system will sweep all the attractor. Each point of the attractor will, sooner or later, be visited by the state of the system. When the Lyapunov exponents have positive real part, the attractor is called strange attractor. Ruelle and Takens gave this definition. The strange attractor has not integer dimensions: it has a fractal structure. Due to the fractal structure of the attractor, invariant in time, two trajectories, initially very close and almost parallel, can diverge drastically and abruptly, though remaining within the attractor.
The logic of the Lyapunov exponents leads to synthesizing the succession of the evolutionary process with an equivalent linear transformation that evolves exponentially and cannot highlight the complexity of the behaviour. We can imagine such a scenario considering a liquid flowing across a permeable medium formed of sand. The trajectories of two fluid particles very close to each other can diverge as soon as they meet a sand particle. The void represents the attractor and constrains the trajectories of the liquid: thus they can take tortuous and long-limbed forms that can be considered something between a volume and a surface or even a line.

7.5. Some final remarks

7.5.1 Consideration on Laplace's *dictum*

The Laplace dictum (*given the precise knowledge of the initial conditions, it should be possible to predict the future of the universe*) has had a meaningful influence on how the deterministic systems (i.e. the systems whose evolution is governed by differential equations) have been considered. Indeed, it is assumed that there is no need to observe the solution for $-\infty < t < \infty$ to see how the deterministic system behaves in the future or in the past. This assumption it is certainly true if the solution of the equations describing deterministic

system is stationary or periodic and can be considered valid even if the solution is quasi-periodic or if a white noise is superimposed on it. The predictability of the future behaviour of a variable whatever can be deduced by the autocorrelation coefficient of that variable: if the autocorrelation coefficient vanishes after a short interval of time, as it happens for turbulent variables, the predictability is only extended to that short interval of time.
Limited predictability is also connected with the strong dependence on the initial conditions: the solutions of a deterministic system originating from very close initial conditions, may not remain close as time goes by as Lorenz has shown.
Moreover, the irregular and disordered behaviour in the evolution of a dependent variable is due to variables not considered or neglected in formulating the laws that govern the process. Deterministic systems are often simplified schematizations of the physical processes they are meant to represent, with an inevitable degree of approximation, whether large or small. Therefore, when the evolution deviates from a regular and orderly behaviour it is expected that this can be due to the variability of some parameter or independent variable that plays a role in the process but it has been neglected in the schematization and that therefore it is not under control.
In conclusion, the future behaviour of a deterministic system can be exactly foreseen, if the initial conditions and all the variables and parameters are exactly known. The unpredictability comes from the approximated knowledge of the initial conditions and from the neglecting of variables and parameters. Even two initial conditions, which differ between them due to the round-off error, cause the completely different evolution of the corresponding originating solutions.

7.5.2 Time horizon and predictability

The behaviour of a deterministic system with positive Lyapunov exponents is unpredictable beyond a well defined time horizon. Suppose we want to predict, with tolerance δ, the value assumed at time t by a variable of the

state of the system whose initial value is measured. The measure is affected by an error ε_0; the error grows with time according to the exponential law:

$$\varepsilon(t) = \varepsilon_0 e^{\lambda_1 t} \tag{7.38}$$

Therefore, $\varepsilon(t)$ will reach the value δ at the time t_p:

$$\delta = \varepsilon_0 e^{\lambda_1 t_p} \tag{7.39}$$

From which:

$$t_p = \frac{1}{\lambda_1} \ln \frac{\delta}{\varepsilon_0} \tag{7.40}$$

In order to increase t_p, ε_0 should be reduced, but the reduction should vary exponentially to be effective. Hence, the order of magnitude of the predictability time is therefore:

$$\theta(t_p) = \frac{1}{\lambda_1} \tag{7.41}$$

7.5.3 SRB measure and Future Perspectives

The attractor can be divided into hyper-volumes and the percentage of time that the state of the system spends in each hyper-volume can be measured. If the time percentages do not vary starting from different initial conditions then it is said that the attractor is endowed with an SRB measure, from the name of the researchers Sinai, Ruelle, Bowen, who proposed it. This measure essentially represents how the probability that the state of the system belong to that hyper-volume is distributed regardless of the initial conditions. This probabilistic vision represents a positive and constructive approach, almost an engineering point of view, which makes the strong dependency on initial conditions more treatable.

A turbulent flow, consisting of many fluid particles, may be considered as a deterministic system, which evolves from the attraction basin to its attractor.

Thus the SRB measure may be regarded as the density of particles per unit of volume, which in turn determines the probability that a fluid particle is in one area rather than another. This point of view offers new perspectives and techniques to be used in investigation on turbulent flows. In particular, it seems possible to use low dimensions deterministic systems able to face successfully the investigation on turbulent flows.

The Fourier decomposition of the velocity vector could be a possible way forward.

At the end of this ride through the chaos a question arises: does the Reynolds approach fit in the light of the achievements obtained by the deterministic theory of chaos? Is the division of the velocity field into ensemble average and fluctuating velocity an obsolete approach today? Probably not. The ensemble average operation requires the repetition of the process for a number of times such as to reach, with sufficient approximation, the limit of the average value if it exists (ergodic process). The repetition of the process ideally presupposes initial conditions that are very close but not exactly identical otherwise the process would repeat itself identically with identical results, i.e. without fluctuations. In Reynolds' approach, initial conditions and their influence is therefore implicitly taken into consideration.

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In this book the fundamentals of turbulent incompressible flows are illustrated, sta rting from considerations on hydrodynamic stability. The characteristics of turbulent flows, the Reynolds-average approach and the turbulent kinetic energy balance are then illustrated. Then most important turbulence models are introduced and described and finally hints on the application of deterministic chaos theory to the study of turbulence are presented.

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