COLLANA
MATHÉMATA 1

ISABELLA FASCITIELLO

FOR THE UNITY OF MATHEMATICS

THE 1954 LANDMARK CONTRIBUTION BY ANDREJ N. KOLMOGOROV TO THE THEORY OF DYNAMICAL SYSTEMS

Roma TrE-Press
2025

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Università degli Studi Roma Tre
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FORTE
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A new branch of which we have desperate need. I call it *paramathematics*. For mathematicians – of the 'theorem proving variety' – only speak about mathematics from within it, with its own criteria of rigor – considering an outside view trivial or irrelevant.
Yet it almost seems to be a direct corollary of Kurt Gödel's great theorems, that mathematics does not possess the language within itself to reflect on its own nature – and how could it, when it cannot even (at the level of one theory) solve the problem of its own consistency.
[Alas, mathematicians live and work in a *terra* that is *incognita* to the population at large, i.e. to everybody else [the *hoi polloi*, the un-mathematical multitudes]].
This is the meaning of *paramathematics*; this is the need for it: for a field, literally, on the side of mathematics, as *metamathematics* is beyond it. This will of necessity be cross- and inter-disciplinary, drawing on mathematics itself but also on logic, philosophy, epistemology, the history of ideas, cognitive psychology, sociology, anthropology, education theory. And, of course, mathematics education.
Of paramathematics we require that it provide mathematics as we know it with context and thus meaning, extra-mathematical meaning and thus criteria, distance, clarity, bird's eye view, integration with thought, history and society.

APOSTOLOS DIOXIADIS
Opening address to the Third Mediterranean Conference of Mathematics Education Athens, January 3, 2003

Un nuovo ramo di cui abbiamo un disperato bisogno. Io lo chiamo *paramatematica*. Perché i matematici – quelli per intenderci che "dimostrano teoremi" – parlano della matematica solo dal suo interno, con i propri criteri di rigore – e considerano una visione dall'esterno banale o irrilevante.
Eppure sembra quasi essere un diretto corollario dei grandi teoremi di Kurt Gödel, che la matematica non possiede in sé il linguaggio per riflettere sulla propria natura – e come potrebbe, se non riesce nemmeno (a livello di una teoria) venire a capo del problema della propria coerenza.
[Ahimè, i matematici vivono e lavorano in una *terra* che è i*ncognita* per le persone in generale, cioè per tutti gli altri [gli *hoi polloi*, le moltitudini non matematiche]].
Questo è il significato della *paramatematica*; questa è la necessità di essa: di un campo, letteralmente, dalla parte della matematica, così come la *metamatematica* è al di là di essa. Un tale campo sarà necessariamente trasversale e interdisciplinare, attingendo alla matematica stessa ma anche alla logica, alla filosofia, all'epistemologia, alla storia delle idee, alla psicologia cognitiva, alla sociologia, all'antropologia, alla teoria dell'educazione. E, naturalmente, alla didattica della matematica.
Alla paramatematica chiediamo che fornisca alla matematica come la conosciamo contesto e quindi significato, significato extramatematico e quindi criteri, distanza, chiarezza, visione a volo d'uccello, integrazione con il pensiero, la storia e la società.

APOSTOLOS DIOXIADIS
Conferenza di apertura della Terza Conferenza Mediterranea di Didattica della Matematica, Atene, 3 gennaio 2003

Mathémata
Collana di studi di paramatematica è promossa dal Dipartimento di Scienze della Formazione e dal Dipartimento di Scienze dell'Università Roma Tre. Accoglie contributi monografici e raccolte in una pluralità di ambiti collegati all'universo della matematica (nella tradizione italiana, le "matematiche complementari"):

- storia delle scienze matematiche (inclusa la storia dell'istruzione e formazione matematica), didattica della matematica, generale e professionale, dall'inizio della scolarizzazione da una prospettiva culturale di inclusione, fino al livello terziario
- epistemologia e filosofia della matematica
- etnomatematica e antropologia della matematica
- comunicazione e museologia della matematica
- letteratura, arti e gioco per la matematica

Una visione allargata della matematica nelle sue molteplici connessioni, che la radicano in esperienze e ambizioni umane e nella cultura, incoraggia la comprensione da parte di un pubblico lettore e fruitore ampio. Perché ognuno di noi ha un rapporto con la matematica, e fin dall'infanzia.

Collana pubblicata nel rispetto del Codice etico dell'Università Roma Tre.

Abstract

The Russian mathematician Andrej Nikolaevich Kolmogorov (1903-1987) delivered a lecture at the International Congress of Mathematicians in Amsterdam on September 9, 1954, titled *The general theory of dynamical systems and classical mechanics*. The lecture's text, published in Russian two years later in the *Congress Proceedings*, presented an innovative research program exploring interrelations among various areas of mathematics. Kolmogorov aimed to develop new approaches to unresolved problems in classical mechanics, drawing on his deep knowledge of 19th-century mathematics – particularly the tradition of mathematical physics.

To his audience, classical mechanics may have seemed outdated compared to the then-dominant theories of relativity and quantum mechanics. However, Kolmogorov firmly believed that classical mechanics was far from obsolete. He argued that Hamiltonian dynamical systems deserved renewed attention, especially from young scholars who could build upon seemingly outdated or neglected theories.

During the first half of the 20th century, the applications of mathematics expanded into a wide range of scientific and technological domains, extending beyond the inanimate to the humanities and life sciences. The dynamical systems approach – what Stephen Smale later called *the modern mathematics of time* – played a significant role in this development. Paradoxically, mechanics as studied through differential equations – a field central to modern science since the 17th century, and particularly to celestial mechanics – had long reached a theoretical impasse.

Kolmogorov's lecture was preceded by two articles he published between late 1953 (shortly after Stalin's death) and the weeks leading up to the congress. The second of these introduced two theorems that would become cornerstones of his research program. These theorems, and the program itself, laid the groundwork for what would become one of the major developments in 20th-century mathematics: KAM theory, named after Kolmogorov, his student Vladimir Igorevich Arnol'd, and the German mathematician Jürgen Moser.

This book explores the origins of Kolmogorov's contribution, from his early fascination with science – particularly astronomy – to his emergence as a central figure in Soviet mathematics and intellectual life. Its aim is to shed

light on the genesis of a theory largely overlooked in the historiography of mathematics, and to examine, through a focused case study, the broader conditions of mathematical research in the 20th century, the evolution of classical mechanics, and the decisions and actions of an individual scholar shaped by a complex personal biography and the historical-political context of his time.

Contents

Introduction

A lecture at the International Congress of Mathematicians, Amsterdam 1954 – and the "true picture of classical mechanics"

THURSDAY, 9th SEPTEMBER, AFTERNOON

2.30-3.30 Concertgebouw, van Baerlestraat

In the chair: J. A. Schouten

A. N. Kolmogorov, Obščaja teorija dinamičeskih sistem i klassičeskaja mehanika (Théorie générale des systèmes dynamiques et mécanique classique).
(Address by invitation of the Organizing Committee).

3.45 Concertgebouw, van Baerlestraat

CLOSING SESSION*)

Fig 1.1. Extract from the program of the International Congress of Mathematicians, Amsterdam, 1954.[1]

> It was a surprise to me that I would have to present a paper at the final session of the Congress in this large hall, which had been known to me rather as a place for the performance of great musical compositions of the world conducted by Mengelberg[2]. The paper which I have prepared, without taking into account that it would occupy such an honourable position in the programme of the Congress, is devoted to a rather special range of problems. My aim is to elucidate ways of applying basic concepts and results in the modern general metrical and spectral theory of dynamical systems to the study of conservative dynamical systems in classical mechanics. However, it seems to me that the subject I have chosen may also be of broader interest, as one of examples of the appearance of new, unexpected and profound relationships between different branches of classical and modern mathematics.
> In his famous address at the Congress in 1900, D. Hilbert said that the unity of mathematics and the impossibility of its division into independent branches stem from the very nature of the science of mathematics.
>
> Andrei N. Kolmogorov's closing speech at the 1954 International Congress of Mathematicians at Amsterdam, printed text in the Congress *Proceedings* (Kolmogorov 1957); English translation in Kolmogorov's *Selected works* (Tikhomirov ed. 1991, p. 355)[3].

[1] (Gerretsent, De Groot 1957, p. 126).

[2] Reference to the Dutch conductor Willem Mengelberg (1871-1951).

[3] See the *Introductory Note to the Bibliography*.

With the above-quoted words, Andrej Nikolaevich Kolmogorov (1903-1987) began his invited plenary lecture at the final session of the International Congress of Mathematicians (ICM) on the afternoon of September 9, 1954, held in the Concert Hall (Concertgebouw) in Amsterdam, Netherlands. It was the second ICM meeting after the interruption caused by the war, and the first at which a delegation from the USSR was present.

The text of the lecture, published in the *Proceedings of the ICM* in 1957 (in Russian, 19 pages; Kolmogorov 1957), is an engaging and elegant example of Kolmogorov's scientific writing. Just as the Hungarian-born American mathematician John von Neumann (1903-1957)[4] had opened the Congress on September 2, Kolmogorov chose to begin by reinforcing a central claim about the unity of mathematics – an idea famously articulated by David Hilbert (1862-1943) in his celebrated address at the Paris ICM in 1900. Kolmogorov blended Hilbert's exhortation to 20th-century mathematicians with a reflection on the particular kind of continuity that characterizes mathematics: a link between past and future, between what he called *classical mathematics* and *modern mathematics*.

The title of Kolmogorov's lecture, *The General Theory of Dynamical Systems and Classical Mechanics*, thus alludes both to the centuries-old tradition of studying motion through differential equations[5] and to the 20th-century theory of dynamical systems – a field rooted in classical mechanics but enriched by new mathematical tools for describing any phenomenon evolving in time.

The adjective "classical" underscores the break between the tradition of *rational mechanics*[6] – from its Newtonian foundations to its reformulations by Joseph-Louis Lagrange (1736-1813) and William Rowan Hamilton (1805-1865) – and the theoretical revolutions of the early 20th century:[7] quantum mechanics and

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[4] (Redei 1999).

[5] For the history of mechanics, see the comprehensive works by (Duhem 1905), (Borel 1943), and (Dugas 1950). For pre-modern mechanics, see (Claggett 1959). See also the essays in Truesdell 1976a, 1976b), and Fraser's review of the research up to the mid-1990s (Fraser 1994). Among more recent contributions, see (Dell'Aglio 1993), (Barrow-Green 1997), (Panza 2003), (Pulte 2003), (Israel 2015), and (Fraser & Nakane 2023). Differential equations were used to describe physical phenomena beyond motion, and mechanics inspired the development of the broader field of mathematical physics in the 18th and 19th centuries.

[6] This expression is uncommon in English; see (Fraser 1994) for further commentary.

[7] In fact, the advent of quantum mechanics – alongside the development of relativity – in the early 20th century marked the emergence of a scientific community of theoretical physicists that became culturally and institutionally distinct from the world of mathematicians

relativity. As Craig Fraser notes in this regard:[8]

«With the establishment of special relativity, it became necessary to introduce the adjective "classical" to delineate the vast range of mechanical doctrines from Newton to Einstein. Classical theories retain their validity and continue to be cultivated extensively today in mathematical engineering. Nevertheless, since Einstein, the classical viewpoint has lost its epistemological primacy as final description of material motion in space and time.» (Fraser 1994, p. 984).

In a recent paper on *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* by Edmund Whittaker (1873-1956) – first published in 1904, with a second edition in 1917 – Severino Collier Coutinho discusses the state of classical mechanics during that period:

«Once a flourishing subject, where a remarkable cross-breeding of mathematics and physics took place, classical mechanics was considered by many to have reached a dead end by the first decades of the twentieth century, except for possible applications to other fields. By the 1950s, some physicists considered classical mechanics as useful only as "the springboard for the various branches of modern physics" and because it afforded "the student an opportunity to master many of the techniques necessary for quantum mechanics" (Goldstein 1950, p. ix).
At about the time that physicists were expressing such thoughts, the tide was already turning in favor of classical mechanics. This was spearheaded by the needs of the age of space exploration and by new advances on the theoretical side. Chief among these were Kolmogorov's results concerning Poincaré's "Problème Général de la Dynamique" (Poincaré 1899, vol. 1, chapter 1, §13, p. 32). Presented at the 1954 International Congress of Mathematicians, these results would give rise to what is now called KAM theory.» (Coutinho 2014, p. 356).

Let us also quote the provocative Clifford Truesdell in the concluding section of his 1976 *History of Classical Mechanics*, titled *On the Decline of Classical Mechanics*:

(Faddeev 1995).

[8] In his previously mentioned contribution on classical mechanics in *The Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, edited by Grattan-Guinness (Fraser 1994).

«The word "classical" has two senses in scientific writing; (1) acknowledged as being of the first rank or authority, and (2) known, elementary, and exhausted ("trivial" in the root meaning of that word). In the twentieth century mechanics based upon the principles and concepts used up to 1900 acquired the adjective "classical" in its second and pejorative sense, largely because of the rise of quantum mechanics and relativity. "Fundamental" in physics came to mean "concerning extremely high velocities, extremely small sizes, or both". Physicists gave less and less attention to classical mechanics because they thought nothing more could be learned from it and nothing new discovered about it, although of course they continued to use it in the design of the experimental apparatus with which they claimed to controvert it. At about the same time "applied" in mathematics came to refer not to the object studied but to the originality and logical standards of the student, again in a pejorative sense.
Engineers still had to be taught classical mechanics, because in terms of it they could understand the machines with which they worked and could devise new machines for new purposes. Research in mechanics came to be slanted toward the needs of engineers and to be carried out largely by university teachers who regarded mathematics as a scullery-maid, not a goddess or even a mistress.» (Truesdell 1976, pp. 127-128).

The exciting new developments in physics stood in sharp contrast to a profound theoretical impasse in classical mechanics. This impasse emerged through the brilliant contributions of the French scholar Jules-Henri Poincaré (1854-1912) to the three-body problem in celestial mechanics at the end of the 19th century.[9] Poincaré recognized that the mathematical structures used to describe natural phenomena often exhibited irregularity – or what would later be called chaos – thereby casting serious doubt on the possibility of predicting their evolution over time.[10]

This theoretical and epistemological crisis in classical mechanics challenged its central role within the mathematical sciences – especially in light of the growing vitality of modern algebra and other branches of mathematics that explored abstract structures independently of their connections to physical phenomena, technological developments, or practical applications (Israel 2015).

[9] In a recent essay on "modern" classical mechanics, Halliwell and Sahakian (2020) remind us that "motions within the Solar System were the most important testing ground for classical mechanics in the first place" (p. xiv).

[10] See (Dahan Dalmenico *et al.* (eds) 1992). On mechanism as an underlying metaphysics of science – highlighting the crucial role of classical mechanics in scientific thought and its subsequent crisis – see (Israel 2015).

The emergence of a new, qualitative theory of differential equations – the theory of dynamical systems, beginning with the seminal contributions of Henri Poincaré and the American mathematician George David Birkhoff (1884-1944) – revealed potential applications beyond classical mechanics. These included phenomena in the life sciences, human social and economic systems, and engineering problems modeled by nonlinear differential equations.[11] Recent studies on the origins of dynamical systems theory have illuminated the diverse research lines that developed during the "decline of classical mechanics," encompassing the rise of mathematical modeling and efforts to apply classical mechanical principles to problems in biology, demography, and economics.

«From Poincaré to the 1960s, the mathematical study of dynamical systems developed in the course of a *longue-durée* history that cannot be unfolded in a cumulative, linear fashion. In particular, this history is not reducible to that of a mathematical theory (which might be called "dynamical systems theory" or the "qualitative theory of differential equations") made by academic mathematicians who would have all contributed a stone to the final edifice. In fact, this history unfolds along various geographic, social, professional, and epistemological axes. It is punctuated by abrupt temporal ruptures and by transfers of methods and conceptual tools. It involves scores of interactions among mathematics, engineering science, and physics along networks of actors with their specific research agendas and contexts. Finally, it is characterized by countless instances of looping back to the past, to Poincaré's work in particular, which are so many occasions for new starts, crucial reconfigurations, and reappreciation of history.» (Aubin, Dahan Dalmedico 2002 pp. 278-279).

This area received sustained attention in the Soviet Union, where Birkhoff's work was further developed in connection with engineering applications – particularly in the field of nonlinear mechanics – as Simon Diner has emphasized:[12]

[11] The origins and development of dynamical systems theory have been the focus of historical research since the 1990s, largely driven by renewed interest in the debate over chaos versus determinism. See (Dahan Dalmedico *et al.* (eds) 1992); (Aubin and Dahan Dalmedico 2002); (Holmes 2007); and (Franceschelli *et al.* (eds) 2007). On Henri Poincaré's contributions, see (Holmes 1990) and (Barrow-Green 1997). For Birkhoff's work, see (Aubin 2005) and (Dell'Aglio 2003). On the topic of stability from Poincaré to Birkhoff, see (Roque 2011).

[12] In his contribution to the volume edited by (Dahan Dalmedico *et al.* 1992), Simon Diner

«[...] le grand public en Occident ignore largement que ce sont essentiellement des savants russes qui ont pendant cinquante ans exploité la partie de l'héritage d'Henri Poincaré, concernant la "théorie qualitative des systèmes dynamiques" et la "mécanique non linéaire" dont le chaos déterministe n'est qu'un des aspects les plus spectaculaires.
Situation créée par la conjonction de l'isolement relatif de l'Union Soviétique et les mobiles internes du développement des mathématiques dans un univers de la physique où la mécanique quantique a ravi la vedette à la mécanique classique. Le langage de Poincaré semblait opaque et ses idées en ont souffert, d'autant plus que les applications qu'il envisageait ne concernaient que l'astronomie.» (Diner 1992, pp. 331-332).

"The problem of integrating systems of differential equations in classical mechanics," Kolmogorov reminded the audience in Amsterdam, had been "a focal point for the mathematics of the 19th century" – as if urging a renewal of ties with the mathematical past. The goal of his lecture was to present a new research program: metrical and spectral methods – specifically, 20th-century measure theory in Hilbert spaces – could shed light on key unresolved issues in classical mechanics, with particular attention to Hamiltonian conservative dynamical systems, which lie at the heart of celestial mechanics.[13]

This area had been somewhat neglected within the mathematical community since the late 1930s, when interest in classical approaches to celestial mechanics began to wane.[14] As such, the topic of his Amsterdam lecture may have appeared surprising or even outdated, addressing a branch of classical mechanics that had received little attention for over fifteen years, both in the Soviet Union and internationally – namely, the study of systems in celestial mechanics, including the critical three-body problem.

In short, in presenting his research to an international audience that had largely moved away from this field, Kolmogorov framed the problem in the following terms:

---

emphasized that the apparent gap between Poincaré's work and Stephen Smale's contributions was not real, but rather a consequence of the "ignorance of the Russian contributions" during the Cold War era and the separation between the West (NATO countries) and the Soviet Union and its Warsaw Pact allies (Diner 1992). See also (Nemytskii 1957).

[13] For a discussion of the meaning of Hamiltonian systems, see the Appendix.

[14] This historical transformation – and the shifting status and relationship between mathematical physics and theoretical physics in the early 20th century – may warrant further investigation.

«For conservative systems, the metrical approach is of basic importance making it possible to study properties of a major part of motions. For this purpose, contemporary general ergodic theory has elaborated a system of notions whose conception is highly convincing from the viewpoint of physics. However, up to now the progress made towards the application of these modern approaches to the analysis of specific problems of classical mechanics has been more than limited. [...] I believe that the time has now come when considerable more rapid progress can be made.» (Kolmogorov 1957, eng. Tr. 1991, pp. 356-357).

Two pivotal theorems had been published by Kolmogorov just a few days before his Amsterdam lecture, on August 31, in a paper appearing in the proceedings of the Soviet Academy of Sciences («Doklady Akademii Nauk SSSR»), titled "On the persistence of conditionally periodic motions under a small change in the Hamilton function" (in Russian; Kolmogorov 1954). These theorems demonstrated the application of new mathematical methods to Hamiltonian conservative dynamical systems and paved the way for further developments. Moreover, they suggested that the impasse noted by Poincaré – particularly regarding the three-body problem in celestial mechanics – could, at least in part, be overcome.

The theoretical significance of Kolmogorov's research program, especially in the context of the evolving status of classical and celestial mechanics within the broader interplay between mathematics and physics, forms the central focus of this book. My aim is to explore its cultural and intellectual origins: why and how Kolmogorov formulated the program, and how the initial theorems he proposed served as foundational elements for its realization.

Scott Dumas has emphasized the key epistemological role of Kolmogorov's contribution in the development of modern science:

«Right from the start, after enunciating his laws of mechanics and gravitation, Isaac Newton ran into difficulties using those laws to describe the motion of three bodies moving under mutual gravitational attraction (the so-called 'three body problem'). For the next two centuries, these difficulties resisted solution, as the best minds in mathematics and physics concentrated on solving other, increasingly complex model systems in classical mechanics (in the abstract mathematical setting, to 'solve' a system means showing that its trajectories move linearly on so-called 'invariant tori'). But toward the end of the 19th century, using his own new methods, Henri Poincaré confronted Newton's difficulties head-on and discovered an astonishing form of ‘unsolvability,’ or chaos, at the heart of the three-body problem. This in turn led to a paradox. According to Poincaré and his followers, most classical systems should be chaotic; yet observers and experimentalists did not see this in nature, and mathematicians working with model systems could not (quite) prove it to be true either. The paradox persisted for more than a half century, until Andrey Kolmogorov unraveled it by announcing that, against all expectation, many of the invariant tori from solvable systems remain intact in chaotic systems. These tori make most systems into hybrids – they are a strange, fractal mixture of regularity and chaos.» (Dumas 2014, preface).

Kolmogorov’s research program led to the development of a new mathematical theory: the so-called KAM theory, named after Kolmogorov and two younger scholars – his former student Vladimir Igorevich Arnol’d (1937-2010) and Jürgen Moser (1928-1999).[15] As Dumas notes:

«It is not a stretch to rank KAM theory alongside the revolutions in modern physics. But KAM theory [...] also had the misfortune of playing out over roughly the same interval during which the revolutions of modern physics took place. Not surprisingly, in that period, physicist abandoned classical mechanics to the few hardy mathematicians who reanimed interested in. The physicists returned with wondrous stories of their exploits in quantum mechanics, relativity, and nuclear physics. *The time has come for mathematicians to tell their tales from this period in a broad setting, too.*» (Dumas 2014, preface, my emphasis).

My investigation was inspired by Dumas’s call to develop a “story of KAM” to address the lack of awareness – both among mathematicians and within the broader scientific community – of a line of research aimed at completing the “true picture of classical mechanics”:

«often thought to have been fully outlined in the 17th century, this picture was not complete until the latter part of the 20th century. Although the mathematical theory is mostly complete, certain applications to problems in physics – especially in celestial and statistical mechanics – have been developed only with great difficulty, and some remain controversial and uncertain even today.» (Dumas 2014, preface, p. 7).

[15] See (Celletti, Froeschlé, and Lega 2003); (Chierchia 2008, 2012); (Dumas 2014, Chapter 4); (Hubbard 2014); and (Diacu and Holmes 1996).

Vladimir Arnol'd was among the first to raise the question of the origins of Kolmogorov's contribution in the 1950s. Having asked this question directly to his former teacher, he later transmitted a testimony regarding a brief conversation in 1984 – thirty years after the Amsterdam lecture (Arnol'd 1997, 2000). This testimony provided a crucial clue and stimulus for my investigation.

Historiography plays a key role in developing a "storytelling" approach that can be shared and internalized as part of the collective understanding of mathematical breakthroughs. Mathematical treatises often obscure the original context of discoveries, presenting achievements in a normalized, logically polished form. My contribution seeks to reconstruct the intellectual and cultural conditions that shaped Kolmogorov's research – a foundational contribution that redefined the status of classical mechanics in the 20th century – and to illuminate the broader cultural context and often difficult paths through which mathematical knowledge is produced.

Reconstructing a Network of Scholars and the Interplay of Disciplines and Methods
Let us begin with Kolmogorov's own words concerning the intellectual origins of his research program. In a note written for the first volume of his *Selected Papers*, published in Moscow in 1985 and edited by Vladimir Mikhailovich Tikhomirov (b. 1934), he explained:

«My papers on classical mechanics appeared under the influence of von Neumann's papers on the spectral theory of dynamical systems and, particularly under the influence of the Bogolyubov-Krylov paper of 1937. I became extremely interested in the question of what ergodic sets (in the sense of Bogolyubov-Krylov) can exist in the dynamical systems of classical mechanics and which of the types of these sets can be of positive measure at present this question still remains open). To accumulate specific information we organized a seminar on the study of individual examples. My ideas concerning this topic and closely related problems aroused wide response among young mathematicians in Moscow.» (Kolmogorov 1991, p. 521).[16]

[16] Kolmogorov's research program on Hamiltonian conservative systems in classical mechanics was the focus of a seminar he led in Moscow during 1957-1958. This seminar gave rise to further research – notably by Vladimir Arnol'd, who worked on the three-body problem, and Yakov Sinai (see §§ 2.1 and 3.3).

Three papers from the 1950s were collected in the *Selected Works* section on classical mechanics, including the published version of Kolmogorov's Amsterdam lecture.[17] Yet Kolmogorov himself noted that these works had been prepared under the influence of earlier contributions published in the 1930s. The first two papers are quite succinct and contain only a couple of references each. In contrast, the more comprehensive overview presented in the published version of the Amsterdam lecture includes 23 bibliographical references, spanning the years 1917 to 1954.[18]

The motivations behind Kolmogorov's intellectual and cultural gesture have been considered by Arnol'd, who described it as an intriguing enigma. A comparison between the authors Kolmogorov mentioned in conversation with Arnol'd, the references cited in his mathematical papers from 1953, 1954, and 1957, and those listed in the 1985 note included in *Selected Works* reveals two distinct groups or networks of scholars.

On one hand, we find figures active in celestial mechanics following Henri Poincaré's seminal *Les méthodes nouvelles de la mécanique céleste* (1892-1899). Notable among them are Carl Vilhelm Ludvig Charlier (1862-1934), author of *Die Mechanik des Himmels* (1902-1907), and Edmund Whittaker (1873-1956), known for his 1899 report on the three-body problem, which later evolved into the classical textbook *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*. These works posed foundational challenges for future research in classical mechanics. In addition, there are younger scholars who engaged with both quantum mechanics and relativity, while still addressing unresolved issues stemming from Poincaré's research – such as Jean-François

[17] The three papers by Kolmogorov, originally published in Russian, are available in English in the reliable translation by Vladimir M. Volosov, published in the 1991 English edition of Volume I of *Kolmogorov's Selected Works* (Tikhomirov 1991). Two of the papers appeared in «Doklady Akademii Nauk SSSR», – the first in November 1953 (Kolmogorov 1953), and the second in August 1954 (the previously mentioned Kolmogorov 1954, in which he states two crucial theorems). The third paper (Kolmogorov 1957) is the version of his lecture at the International Congress of Mathematicians in Amsterdam, published three years later in the *Proceedings of the Congress*, edited by Johan C.H. Gerretsen and Johannes De Groot (Gerretsen and De Groot 1957). Additional English translations and one French version of the Amsterdam lecture are also available (see Introductory Note to the Bibliography).

[18] The list includes Kolmogorov (1953, 1954). The earliest reference is to Émile Borel's *Leçons sur les fonctions monogènes uniformes d'une variable complexe* (1917), and the most recent is to the 1954 paper by the Soviet mathematician Mstislav Igorevich Grabar (1925-2006), "On Strongly Ergodic Dynamical Systems."

Chazy (1882-1955) in France, and Otto Yulyevich Schmidt (1891-1956), a polymathic Soviet scholar, close to Kolmogorov, and editor of the *Great Soviet Encyclopedia*.[19]

On the other hand, there is a more tightly connected group of scholars associated with George Birkhoff's (1884-1944) emerging theory of dynamical systems. Between 1931 and 1937, this network explored Hamiltonian systems in classical mechanics using Hilbert spaces and measure theory – an approach that played a crucial role in the development of modern ergodic theory. A foundational contribution in this area was Bernard O. Koopman's (1900-1981) 1931 paper, "Hamiltonian Systems and Transformations in Hilbert Space" (Koopman 1931).

Let us now turn to Kolmogorov's own words on this development:

«After the work of H. Poincaré, the fundamental role of topology for this range of problems became clear. On the other hand, the Poincaré-Carathéodory recurrence theorem initiated the "metrical" theory of dynamical systems in the sense of the study of properties of motions holding for "almost all" initial states of the system. This gave rise to the "ergodic theory", which was generalized in different ways and became an independent centre of attraction and a point of interlacing for methods and problems of various most recent branches of mathematics (abstract measure theory, the theory of groups of linear operators in Hilbert and other infinite-dimensional spaces, the theory of random processes, etc.).» (Kolmogorov 1957, eng. tr. 1991, pp. 355-356).

The variety of references suggests that, in the early decades of the 20th century, the young Kolmogorov closely followed contemporary developments both in classical mechanics – particularly the central question of the three-body problem in celestial mechanics – and in Birkhoff's emerging approach to a general theory of dynamical systems, grounded in the qualitative analysis of differential equations.

In Chapter 1, I examine the mathematical landscape that forms the backdrop to Kolmogorov's research program and the initial development of KAM theory. This includes a discussion of key aspects in the evolution of classical mechanics and the general theory of dynamical systems during a transitional period in the history of mechanics, spanning the late 19th and early 20th centuries.

[19] See (Chazy 1929, 1932) and (Schmidt 1947).

Andrej N. Kolmogorov in Soviet Science and His Contribution to Classical Mechanics

Understanding the origins of Kolmogorov's seminal contribution – his identification of new directions for addressing long-standing problems in classical mechanics – has led me to explore key aspects of the intellectual and cultural trajectory of this exceptional 20th-century scholar. Kolmogorov confronted a period marked by the radical transformation of the relationship between physics and mathematics, and by the changing status of classical mechanics, long considered the core of modern science. He lived through an era of profound upheaval in the former Russian Empire, including the development of a rich network of scientific schools and the growing entanglement of science, technology, the state, and civil society.

Kolmogorov's contributions and research program in classical mechanics are deeply rooted in his scientific biography and the broader context of Soviet science.[20] As he told Arnol'd, "he had been thinking about this problem for decades, starting from his childhood" (Arnol'd 1997, p. 1). His former student Yakov Grigorevich Sinai similarly observed that "apparently the interests of Kolmogorov in ergodic theory had already started in the 1930s" (Sinai 1989, p. 833).

Among Soviet researchers in celestial mechanics were the aforementioned Otto Yulyevich Schmidt, as well as Boris Vasilyevich Numerov (1891-1941?), a leading figure in the flourishing Soviet astronomical community supported by a network of observatories. Birkhoff's approach was followed by prominent Soviet scholars, notably Nikolay M. Krylov (1879-1955) in Kyiv, who worked in nonlinear mechanics, and Vyacheslav Vasil'evich Stepanov (1889-1950) in Moscow, who in 1930 initiated a seminar on the qualitative theory of differential equations – attended, among others, by Kolmogorov (Nemytskii 1957).[21] "Stepanov was among the first in our country to under-

[20] Following Kolmogorov's death, his former student Vladimir Mikhailovich Tikhomirov (b. 1934) wrote a brief but insightful biographical essay titled *The Life and Work of Andrei Nikolaevich Kolmogorov* (Tikhomirov 1988). For further references on the literature available on Kolmogorov, see the Introductory Note to the Bibliography.

[21] Nikolai Luzin was the dominant figure in Moscow mathematics at the time; however, the British statistician David G. Kendall (1918-2007) offered a different perspective in his remembrance of Kolmogorov: «A number of mathematicians stimulated Kolmogorov's earliest mathematical research, but perhaps his principal teacher was Stepanov. In 1922 Kolmogorov produced a synthesis of the French and Russian work on the descriptive theory

stand the significance of the metric theory of general dynamical systems begun in the works of Poincaré and Birkhoff, and he made an essential contribution to it" (Myshkis, Oleinik 1990, p. 180). Stepanov authored a textbook on differential equations published in 1936, and a second textbook – co-authored with Viktor Vladimirovich Nemytskii (1900-1967) – on the qualitative theory of differential equations, first published in 1947.[22] Notably, this latter work is the only bibliographical reference cited in Kolmogorov (1953). Furthermore, (Sinai 1989) emphasizes that the works of von Neumann were being followed closely in the Soviet Union during the 1930s.

Some papers by Kolmogorov dating back to the 1930s reveal his early interest in the general theory of dynamical systems and in ergodic theory. However, the years 1936-1937 marked a sharp intensification of Stalin's purges in the Soviet Union, including a massive campaign against astronomers. A dark period began – culminating with the USSR's entry into the Second World War in 1941 – and only came to a close with Stalin's death in early 1953. The international scientific connections of scholars from the Russian Empire were severely disrupted. During these years, Kolmogorov appears to have conducted quiet, unpublished work on Hamiltonian systems relevant to the unresolved issues in classical mechanics. As I will argue, his publications from the 1950s carried not only scientific significance but also deep cultural and, arguably, political meaning during a time of reconstruction and – perhaps – renewed hope.

Chapter 2 focuses on the biographical and cultural factors that shaped Kolmogorov's education and intellectual interests, drawing on the growing body of literature concerning the evolution of science and scientific education in the Russian Empire, from the Tsarist regime to the Soviet Union. Kolmogorov's engagement in scientific and epistemological debates helps illuminate his views on the role of mathematics in the study of natural phenomena and his enduring commitment to classical mechanics. The chapter also explores the impact of Stalin's purges on the field of astronomy and the prevailing emphasis on dissipative systems in technological applications.

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of sets of points, and at about the same time he was introduced to Fourier series in Stepanov's seminar. This was when he made his first mathematical discovery – that there is no such thing as a slowest possible rate of convergence to zero for the Fourier coefficients of an integrable function.» (Kendall 1991, p. 303).

[22] Second edition published in 1949; translated into English in 1960 by Princeton University Press.

These pressures may have led Kolmogorov to carry out his research on celestial mechanics discreetly during certain periods.

Building on the reconstruction of the mathematical landscape behind Kolmogorov's work in classical mechanics and dynamical systems, and the contextual elements of his scientific biography, Chapter 3 examines the formulation and proof of his theorem on the persistence of invariant tori in Hamiltonian systems, as presented in (Kolmogorov 1954). This includes an analysis of the Diophantine condition, which is central to the theorem's proof, and a comparison with its use in a 1942 paper by the German mathematician Carl Ludwig Siegel (Ghys 2004). Through this historical analysis, I explore the significance of Kolmogorov's 1954 theorem, along with the lesser-known second theorem in the same article, which concerns the measure of the set of persistent tori in phase space. Both results are framed within Kolmogorov's broader research program in classical mechanics, as outlined in his Amsterdam lecture. It was within this program that Arnol'd would later apply Kolmogorov's methods to the three-body problem in celestial mechanics (Arnol'd 1963b, 2009).

Finally, I consider aspects of the transmission of Kolmogorov's research program – shaped in part by the Cold War context of international scientific relations – and some of the misunderstandings concerning the balance between the formulation of goals and methods, and the standards of mathematical demonstration. Deepening our understanding of Kolmogorov's seminal work from 1953-54 can thus shed light on the evolution of mathematical thought in the 20th century, particularly in its relation to science as a means of investigating the natural world.

My research has greatly benefited from the advice and insights of many scholars in Italy and abroad. I would like to thank my PhD supervisors at Roma Tre University, Luca Biasco and Ana Millán Gasca, for their invaluable guidance, suggestions, and joint discussions on key historiographical and mathematical questions. I am also grateful to the Department of Mathematics and Physics at Roma Tre University, where I was able to conduct my research in a stimulating and supportive environment. My sincere thanks go to Luigi Chierchia and Michela Procesi, as well as Jessica Elisa Massetti and Shulamit Terracina.

I am especially thankful to Efthymios Nicolaïdis for his insights into astronomy in the Soviet Union, and to Alexander Karp for generously sharing his deep knowledge of the history of mathematics and mathematics education in the Russian Empire. I had the privilege of engaging with him during and after his visit to the Department of Education at Roma Tre University.

I also thank Luca Dell'Aglio for his comments and support on historiographical methodology. Special thanks go to Yakov Sinai for his personal testimony, and to Alfonso Sorrentino for helping to arrange contact with him. I am also grateful to Paola Magrone and Ana Millán Gasca for their support in preparing this research for publication in *Mathemata*.

To my family, and to my husband Antonio, I express my deepest affection and gratitude for their presence and constant encouragement. This book is dedicated to our daughter Elisa.

Chronology[23]

1903, April 25 – Birth of Andrej Nikolaevich Kolmogorov. His mother died in childbirth.
1906-1910 – Homeschooling under the supervision of his aunts at his maternal grandfather's estate in Tunoshna (near Yaroslavl).
1910-1917 – Student at the Evgenja Albertovna Repman private school in Moscow.
c. 1919 – Death of his father.
1920 – Enrolment at Moscow University (mathematics) and the D. I. Mendeleev Institute of Chemical Engineering (metallurgy).
1922-1925 – Mathematics and physics teacher at the Potylikhin Experimental School in Moscow.
1924, January 21 – Death of Lenin.
1925 – Graduation from Moscow University and beginning of postgraduate work.
1927 – *Dynamical Systems* by George David Birkhoff (1884-1944).
1929 – Researcher at the Institute of Mathematics and Mechanics, Moscow University. Engages in contemporary debates on the nature of mathematics.
1929-1930 – Attends Vyacheslav Vassilievich Stepanov's seminar on the qualitative theory of differential equations at Moscow University.
1930, June - 1931, May – Academic journey to Germany and France with Pavel S. Aleksandrov (1896-1982).
1931 – "Hamiltonian Systems and Transformations in Hilbert Space" by Bernard Osgood Koopman (1900-1981).
1932 – "Zur Operatorenmethode in der klassischen Mechanik" by John von Neumann (1903-1957).
1932 – Proof of the quasi-ergodic hypothesis by von Neumann.
1932 – "Dynamical Systems of Continuous Spectra" by Koopman.
1933 – Proof of the quasi-ergodic hypothesis by Birkhoff.
1933 – *Grundbegriffe der Wahrscheinlichkeitsrechnung* (Russian edition in 1936).
1936 – Luzin placed under scrutiny at the Academy of Sciences.
1936, October 20 – Arrest of Boris V. Numerov (1891-1941?).
1937 – "La théorie générale de la mesure dans son application à l'étude des systèmes dynamiques de la mécanique non linéaire" by Krylov and Bogoliubov.
1937 – "A Simplified Proof of the Birkhoff-Khinchin Ergodic Theorem" by Kolmogorov.
1938 – Article "Mathematics" by Kolmogorov in the *Great Soviet Encyclopedia*.

[23] Kolmogorov's papers were originally published in Russian.

1939 – Elected member of the Academy of Sciences.

1942 – "Iteration of Analytic Functions" by Carl Ludwig Siegel (1896-1981).

1942 – Marriage to Anna Dmitrievna Egorova.

1953, March 5 – Death of Stalin.

1953, November 13 – "On Dynamical Systems with an Integral Invariant on the Torus" by Kolmogorov.

1954, August 31 – "On the Preservation of Conditionally Periodic Motions under Small Variations of the Hamilton Function" by Kolmogorov.

1954, September 9 – Plenary lecture at the ICM in Amsterdam: "The General Theory of Dynamical Systems and Classical Mechanics".

1957 – Publication of Kolmogorov's 1954 lecture in the *Proceedings of the International Congress of Mathematicians*, Amsterdam.

1957, Autumn – Ph.D. course on the theory of dynamical systems in Moscow, attended by Vladimir Arnol'd and Yakov Sinai.

1958, March 22 – Talk in Paris at the Analytical Mechanics and Celestial Mechanics Seminar, led by Maurice Janet (1888-1983).

1959 – Arnol'd defends his dissertation under Kolmogorov's supervision.

1962 – "On Invariant Curves of Area-Preserving Mappings of an Annulus" by Jürgen Kurt Moser (1928-1999).

1963 – "Proof of a Theorem of A. N. Kolmogorov on the Preservation of Conditionally Periodic Motions under a Small Change in the Hamilton Function" (Arnol'd, in Russian).

1963 – *Foundations of Mechanics* by Ralph H. Abraham (1936-2024).

c. 1984 – Brief conversation between Kolmogorov and Arnol'd about the origins of Kolmogorov's work on invariant tori.

1985 – Short note by Kolmogorov on his papers on classical mechanics published in Volume 1 of his *Selected Works* (in Russian).

1 The mathematical landscape

> The branch of physics known as "classical mechanics" originated in the seventeenth century, but wasn't called that until the discovery of quantum mechanics in the 1920s. It was quantum mechanics that most profoundly changed our understanding of how and why particles move as they do, and even what a particle is. Quantum mechanics was so completely different that the word "classical" had to be added to the older theory to make it clear which mechanics was meant. At the same time, quantum mechanics was heavily inspired by the formulations of classical mechanics by Lagrange and Hamilton dating back to the eighteenth and nineteenth centuries.
> In many situations, using quantum mechanics and/or relativity to study a physical system would be tantamount to shooting a fly with a catapult. Roughly speaking, classical mechanics works very well (i.e., agrees with experiments) for macroscopic objects that are moving at speeds much less than the speed of light, and where gravity is not too strong – and also where our experimental measurements are not too precise.
> Take the motions of the planets around the sun and moons round their planets, for example. Motions with the solar system were the most important testing ground for classical mechanics in the first place, and for nearly all purposes classical mechanics in this domain works as well now as it ever did.
>
> Thomas M. Helliwell, Vatche V. Sahakian, *Modern Classical Mechanics* (2020), Preface, xiii-xiv.

At the end of the 19th century, the research of the French scholar Henri Poincaré (1854-1912) demonstrated to the international scientific community that new mathematical tools and conceptual frameworks – so to speak, "new clothes" for old problems – could offer fresh theoretical perspectives on the study of mechanical phenomena in celestial motion, which, as Helliwell and Sahakian put it in *Modern Classical Mechanics* (2020), was "the most important testing ground for classical mechanics" (pp. xiii-xiv).

In the history of mechanics,[1] there has been a steady evolution in mathematical tools, shaped by a dynamic tension between physical objects and mathematical relations. Physical objects initially served as the principal moti-

[1] See (Dugas 1955), (Clagett 1959), (Truesdell 1968).

vation for seeking improvements and new approaches, while mathematical relations – developed in the effort to understand and predict natural phenomena of motion and stability – gradually acquired an autonomous existence, independent of their original physical foundations.

Today, fields such as abstract algebra, topology, functional analysis, and probability theory exist as autonomous branches of mathematics. Yet this current independence can easily obscure the depth of their historical connections to the open problems of mechanics, particularly those in celestial mechanics, beginning with the classical three-body problem.

The early decades of the 20th century were a fruitful period for research in both physics and mathematics, marked by disciplinary restructuring. The foundational developments of quantum mechanics and the theory of relativity were accompanied by significant, though often underexplored, efforts to rethink and revitalize 19th-century mechanical studies – what was increasingly coming to be known as classical mechanics. Celestial mechanics retained a powerful allure for late 19th-century scholars, stemming not only from Isaac Newton's groundbreaking work at the heart of modern mathematical physics, but also from a timeless human aspiration to understand the positions and motions of the stars across the celestial sphere (Diacu, Holmes 1996; Wilson 1994).

With his three-volume treatise *Les méthodes nouvelles de la mécanique céleste* (1892–1899), Henri Poincaré opened a new path. His qualitative analysis of differential equations not only introduced innovative mathematical methods for studying classical dynamics, but also laid the foundation for a new area of research: the general theory of dynamical systems. This mathematical theory, though abstract and distant from its physical roots, would prove applicable to a wide array of phenomena involving temporal evolution across diverse scientific fields.

It is within this dual framework – classical mechanics and the general theory of dynamical systems – that Kolmogorov would present his research program during his closing lecture at the 1954 International Congress of Mathematicians in Amsterdam. In this chapter, I examine the mathematical landscape of the 1920s and 1930s in both of these areas, which formed the background to Kolmogorov's proposal:

«I had thought for a long time about problems in celestial mechanics, from childhood, from Flammarion, and then – reading Charlier, Birkhoff, the mechanics of Whittaker, the work of Krylov and Bogolyubov, Chazy, Schmidt. I had tried several times, without results. But here was a beginning.» (Kolmogorov, quoted in Arnol'd 2000, p. 90).

I begin with a concise overview of research on the three-body problem – one of the initial sources of inspiration for Hamilton's own contributions – during the late 19th century. In the second part of the chapter, I examine an informal research network that connected scholars in the USSR and the USA during the 1930s. This network, inspired by George David Birkhoff's pioneering work rooted in Poincaré's contributions, was also explicitly acknowledged by Kolmogorov in the note included in the first volume of his *Selected Works*, published in Moscow in 1985.

1.1 Between past and future: celestial mechanics at the turn of the 20th Century

> «ASTRONOMY is not only one of the most ancient of the physical sciences, but also one of those which present the most alluring invitations to the contemplative mind. The starry heavens, spangling with countless luminaries of every shade of brilliancy, and revolving in eternal harmony round the earth, constitute one of the most imposing spectacles which nature offers to our observation. The waning of the placid moon, the variety and splendour of the constellations, and the dazzling lustre of the morning and evening stars, must in all ages have excited emotions of admiration and delight.»
>
> Robert Grant, *History of Physical Astronomy: From the Earliest Ages to the Middle of the 19th Century, Comprehending a Detailed Account of the Establishment of the Theory of Gravitation by Newton, and Its Development by His Successors, with an Exposition of the Progress of Research on All the Other Subjects of Celestial Physics* (1852), p. i.

The stars and planets – those tiny points scarcely visible to the naked eye – have long appeared to move undisturbed across the sky, capturing the imagination of diverse peoples and cultures who pondered their nature and motion. The apparent regularity in the movement of celestial bodies, often ascribed to divine or supernatural influence, became an object of study from the earliest civilizations, as part of an arithmetical quest to predict lunar and planetary phenomena. Interest in astronomical events stemmed from multiple motivations: most notably, knowledge of celestial periodicity provided insight into seasonal cycles, essential for agriculture. Yet, alongside this practical concern was a profound human desire to understand the visible world.

The systematization of mechanics into a unified framework, culminating in Newton's formulation of the law of universal gravitation, brought new depth to the mechanical study of celestial bodies. The planets of the solar system, being approximately spherical and small in size relative to the vast distances separating them, could be modeled as point masses, allowing Newton's laws to be applied to their dynamics.

If only the interaction between the Sun and each individual planet were considered, the resulting motion would be an elliptical orbit around the Sun, which occupies one of the foci – an outcome described by Kepler's laws. This scenario defines the so-called "two-body problem," initially solved geometrically by Newton. A more rigorous mathematical treatment, however, was later developed by Swiss mathematicians Johann Bernoulli (1667-1748) and Leonhard Euler (1707-1783).

Solved, in the context of classical mechanics, means that the system of differential equations governing the two-body problem has been shown to be integrable – that is, it admits a sufficient number of independent conserved quantities (or first integrals) to allow for a complete analytical solution. In this case, the key physical parameters of the system – such as the semi-major axis and the eccentricity of the elliptical orbit – remain constant over time or, in mathematical terms, are constants of motion.

The two-body problem. The two-body problem can be schematized as two material points moving in a three-dimensional Euclidean space; each point is therefore identified by three coordinates and, for this reason, the problem has 6 degrees of freedom. If we call $x_1=(x_{11}, x_{12}, x_{13})$ and $x_2=(x_{21}, x_{22}, x_{23})$ the spatial coordinates of the two bodies 1 and 2, m_1 and m_2 the two masses, and F_1 and F_2 the forces acting on bodies 1 and 2, respectively, then the equations of motion are

$$\begin{cases} m_1 \frac{d^2 x_1}{dt^2} = F_1(|x_1 - x_2|) \\ m_2 \frac{d^2 x_2}{dt^2} = F_2(|x_1 - x_2|) \end{cases}$$

For its resolution, the problem is shown to reduce to two decoupled problems: one is a trivial uniform rectilinear motion, and the other becomes a two-degree-of-freedom problem. This results in a system of two ordinary differential equations in two unknowns, one of which depends on a single variable, allowing the solution to be found.

In the Solar System, however, not only do planet-Sun interactions play a role, but also, albeit with lower intensity, planet-planet interactions and, additionally, interactions between a planet and its satellites. These forces perturb the elliptical orbits described by individual planets, and although the effect is slow, catastrophic scenarios over very long time periods – such as the collision of two planets or the ejection of a planet from its orbit – cannot be ruled out a priori.

An integrable system, such as the two-body problem in which the equations of motion can be solved exactly, is thus altered by small perturbations arising from other gravitational interactions, making the resulting problem generally non-integrable. The two-body problem is one of the few cases of integrable systems, represented by a system of equations describing the motion of two planets, which can be solved exactly. Other examples include one-dimensional Hamiltonian systems, such as the harmonic oscillator and the simple pendulum, the so-called Lagrange top, the Kovalevskaya top, geodesic motion on an ellipsoidal surface, and others.

By contrast, the motion of a planet in the Solar System, when considering gravitational interactions with other planets or celestial bodies, becomes a non-integrable problem that is practically intractable from a mathematical perspective. The planets are in constant motion, their relative positions change over time, and the forces they exert on one another continuously vary in both direction and intensity throughout their orbital motion. If these forces were to balance each other out, the planets would remain in the same elliptical orbits observed by Newton indefinitely. Otherwise, significant deviations from their expected orbital motion could occur.

Mathematically solving a differential equation problem with such a large number of variables becomes extremely complex.

However, it is important to note that planet-planet and planet-satellite interactions are negligible compared to the interactions between planets and the Sun. This is because gravitational attraction depends on the masses of the interacting bodies, and planetary masses are significantly smaller than that of the Sun.[2]

Perturbation theory deals with problems in which a small parameter appears ("small" in the sense that it is constrained not to exceed a certain threshold value). In the three-body problem involving the Earth, Sun, and Mars, this parameter is represented by the small ratio between the mass of Mars and the mass of the Sun. The parameter characterizes the difference between the system under study and a similar, "nearby" system that can be integrated exactly.

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[2] The mass of the planets is about one-thousandth part of the mass of the Sun.

Thus, we are dealing with systems – typically written in the Hamiltonian formalism – that deviate only slightly from an integrable system. Perturbation theory, initially addressed by Newton from a geometric perspective, became a central topic in celestial mechanics starting in the second half of the 18th century, with contributions from Laplace and Lagrange, followed by Charles Eugène Delaunay (1816-1872)[3] and Urbain Le Verrier (1811-1877).

This provided the starting point for Poincaré's work at the end of the 19th century.

Poincaré built upon the existing studies of the Finnish astronomer Johan August Hugo Gyldén (1841-1896), who worked at the Pulkovo and Stockholm observatories. Gyldén's objective was to use perturbation theory to derive mathematical series that could describe planetary orbits over arbitrarily long time periods. "[…] le savant qui a rendu à cette branche de l'Astronomie les services les plus éminents est sans contredit M. Gyldén," Poincaré wrote in the introduction to the first volume of *Les méthodes nouvelles de la mécanique céleste* (Poincaré 1892-99). In this way, it would be possible to answer the fundamental question of whether the Solar System is stable or not (Markkanen 2007, Bohlin 1897).

Let us now examine in greater detail how perturbation theory applies to the problem of planetary motion in the Solar System.

As previously mentioned, in a first approximation, planet-planet interactions could be neglected due to their relative insignificance. We can therefore begin by considering the system of differential equations that account only for the interactions between individual planets and the Sun. This problem is integrable: as in the two-body problem, planets describe elliptical orbits around the Sun for infinite time. However, if we no longer neglect the smaller interactions, the orbits will undergo variations. This leads to what is known as the n-body problem, where n represents the number of celestial bodies interacting with one another.

The so-called n-body problem, with $n \geq 3$, is therefore considerably more complex than its reduction to just two bodies – it is enough to consider that for $n = 3$ there is not a general solution (Barrow-Green 1997, Marcolongo 1915, Whittaker 1899). In his *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies; with an Introduction on the Problem of Three Bodies*

[3] Delaunay developed a theory of the motion of the Moon based on the theory of perturbations (Delaunay 1860-67). See §1.4.2.

(Whittaker 1917), first published in 1904, the English mathematician Edmund Taylor Whittaker (1873-1956) introduces chapter XIII, *The Reduction of the Three-Body Problem*, defining the problem as "the most celebrated of all dynamical problems":

«*The most celebrated of all dynamical problems* is known as the Problem of Three Bodies, and may be enunciated as follows: Three particles attract each other according to the Newtonian law, so that between each pair of particles there is an attractive force which is proportional to the product of the masses of the particles and the inverse square of their distance apart: they are free to move in space, and are initially supposed to be moving in any given manner; to determine their subsequent motion. The practical importance of this problem arises from its applications to Celestial Mechanics: the bodies which constitute the solar system attract each other according to the Newtonian law, and (as they have approximately the form of spheres, whose dimensions are very small compared with the distances which separate them) it is usual to consider the problem of determining their motion in an ideal form, in which the bodies are replaced by particles of masses equal to the masses of the respective bodies and occupying the positions of their centres of gravity. The problem of three bodies cannot be solved in finite terms by means of any of the functions at present known to analysis. This difficulty has stimulated research to such an extent, that since the year 1750 over 800 memoirs, many of them bearing the names of the greatest mathematicians, have been published on the subject.» (Whittaker 1917, p. 339, my emphasis).

To describe the problem in terms of differential equations, we can use the notation adopted by Whittaker himself in (Whittaker 1917).

The three-body problem: Let m_1, m_2 and m_3 be the masses of three bodies and r_{23}, r_{13} and r_{12} the reciprocal distances between them. Given an orthogonal system of Cartesian axes O_{xyz}, we can denote the coordinates of the positions of the three masses with respect to it as (q_{11}, q_{12}, q_{13}), (q_{21}, q_{22}, q_{23}) and (q_{31}, q_{32}, q_{33}).
The force of attraction between two masses *mi* and *mj* is $F = km_i m_j r^{-2}$, with k is a constant and, with a suitable choice of units, we can assume $k = 1$.
The kinetic energy and potential energy of the system of three

mutually attracting masses are, respectively:

$$T = \frac{1}{2}\sum_{i=1}^{3} m_i(\dot{q}_{i1}^2 + \dot{q}_{i2}^2 + \dot{q}_{i3}^2)$$

and

$$V = -\frac{m_2 m_3}{r_{23}} - \frac{m_1 m_3}{r_{13}} - \frac{m_1 m_2}{r_{12}}.$$

Thus the equations of motion of the system formed by the three bodies are:

$$m_i \ddot{q}_{ij} = -\frac{\partial V}{\partial q_{ij}} \qquad i, j = 1, 2, 3$$

These are nine second-order differential equations, and therefore the system is of order 18.
Lagrange would later show that this system can be reduced to a sixth-order system[a].
If we want to write the equations in Hamiltonian form, we can denote:

$$H = \sum_{i,j=1}^{3} \frac{p_{ij}^2}{2m_i} + V$$

where $p_{ij} = m_i \dot{q}_{ij}$ denotes the j-th component of the momentum of the mass body m_i. So, the equations of the three-body system are:

$$\frac{dq_{ij}}{dt} = \frac{\partial H}{\partial p_{ij}}, \qquad \frac{dp_{ij}}{dt} = -\frac{\partial H}{\partial q_{ij}},$$

with $i, j = 1, 2, 3$.

[a] See (Whittaker 1917, pp. 338-355).

Is it possible to explicitly find its general solution for all times? One way to prove the integrability of the problem – and therefore its complete resolution – is by searching for uniform integrals: a uniform (or prime) integral for a problem defined by a system of differential equations can be defined as a function that remains constant along the solutions of the system.

For example, the total energy of the system is a uniform integral in the three-body problem, because it remains constant.

The existence of a number of independent prime integrals equal to the order of the system of differential equations (i.e., the number of degrees of freedom of the problem) implies the integrability of the problem.

Therefore, if we wanted to prove the solvability of the three-body problem through the existence of uniform integrals, we would need to find eighteen of them, each independent of the others:

«[...] the problem of three bodies possesses 10 known integrals: namely the six integrals of motion of the center of gravity, the three integrals of angular momentum, and the integral of energy; these are generally called the *classical* integrals of the problem.» (Whittaker 1917, p. 358).

The German mathematician and astronomer Ernst Heinrich Bruns (1848-1919), in an 1887 paper (Bruns 1887), demonstrated that for the general three-body problem there are no uniform integrals beyond the classical ones.[4]

Given these difficulties, the study of the system can be reduced to a simpler case: the so-called *restricted three-body problem*, in which a particle of negligible mass moves under the attraction of two other bodies of positive mass rotating in circles about their center of gravity.

Nevertheless, shortly after Bruns' paper, Poincaré formulated in his *Mémoire* (Poincaré 1890) – and subsequently in the first volume of (Poincaré 1892-99) – an extension of Bruns' theorem, proving the non-existence of uniform integrals also for the restricted three-body problem. Therefore, the integrability problem cannot be addressed by going this route. Another approach is to use perturbation theory.

The solutions of the equations of motion can be described by means of formal power series that depend on the perturbation which deviates the problem from the closest integrable one. In addition to Gyldén's contribution, there was another Swedish scientist, Anders Lindstedt (1854-1939), who developed one of the series of perturbations that describe the solutions – still widely used in celestial mechanics today.

While Gyldén was an astronomer with a strong theoretical bias, Lindstedt combined theoretical interest in the three-body problem with

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[4] See (Whittaker 1899, pp. 157-159).

practical applications. The Lindstedt series was the method most used by Poincaré and by his successors, including Kolmogorov.

The main issue was then to study the convergence of these series which, in most cases, appeared to be divergent. The reason for the lack of convergence was due to disturbances caused by the so-called *small denominators* (or *small divisors*). In fact, the construction of the series implies that within the coefficients of the terms there are denominators that can be zero or dangerously close to zero, causing the coefficients to tend toward infinity and, therefore, making the series itself diverge.

These denominators take the form of linear combinations of the frequencies of non-perturbed motions with integers, of the type:

$$m_1\omega_1 + m_2\omega_2 + \ldots m_n\omega_n,$$

where ω_i, $i = 1 \ldots n$ are real numbers representing the frequencies of the planets, and $m_1, \ldots, m_n$ are integers.

If the ratios of the frequencies are a rational numbers, these denominators can cancel out. This situation is nowadays described as exact resonance between the planets: after a certain number of periods, the initial configuration of their mutual positions repeats itself.

In the vicinity of a resonance – i.e. when the frequencies are close to being commensurable – the small divisors remain very close to zero and, in general, it is not possible to predict the dynamic effects that follow. The repetition of near-identical configurations amplifies the perturbation effect and, in many cases, causes the instability of the resonant orbit.

An example: the frequencies of the motions of Saturn and Jupiter. In their motion of revolution around the Sun, Saturn and Jupiter move with frequencies equal, respectively, to approximately

$$\omega_S = 120'' \qquad and \qquad \omega_J = 299''$$

The two frequencies are almost commensurable, since

$$5\omega_S - 2\omega_J \approx 0$$

Now, the series that describes the motion of the two planets, derived from perturbation theory, is of the type

$$\sum_{m,n\neq 0} \frac{a_{nm}}{n\omega_S + m\omega_J} e^{i(n\omega_S + m\omega_J)t}$$

and, therefore, we find in the denominator a quantity that, for infinitely interger values of n and m, is close to 0.

Difficulties due to small denominators accompanied the theories of celestial mechanics throughout the first half of the 20th century. We will see in Chapter 3 that Kolmogorov overcame this problem by adopting a necessary condition on the ratio between the frequencies of the motions.

1.1.1 "Properties holding for almost all the initial states of the system": Henri Poincaré (1854-1912) recurrence theorem (1890) towards a metrical approach to dynamical systems

One hundred years before the International Congress of Mathematicians in Amsterdam in 1954, Jules Henri Poincaré was born in Nancy. A pioneer in the use of algebraic geometry and topology in the study of celestial mechanics, his work contributed significantly to the study of the three-body problem by introducing a new approach known as the "qualitative study of differential equations." The term *qualitative* refers to the study of the behavior of the solutions of a system of differential equations, obtained through a geometric approach, without requiring an explicit expression for those solutions. This proved especially necessary in celestial mechanics, where, as we have seen, it was not possible to determine the solutions of the system explicitly.

This innovation finds its fullest expression in the three volumes *Les méthodes nouvelles de la mécanique céleste* (Poincaré 1892-99), though the origins of his research in this field date back more than ten years earlier, to his "Mémoire sur les courbes définies par une équation différentielle (I)" (Poincaré 1881), in which the author himself introduced the adjective *qualitative* in reference to the geometric study of the curve defined by the function under consideration:

«Une théorie complète des fonctions définies par les équations différentielles serait d'une grande utilité dans un grand nombre de questions de Mathématiques pures ou de Mécanique. Malheureusement, il est évident que dans la grande généralité des cas qui se présentent on ne peut intégrer ces équations à l'aide des fonctions déjà connues, par exemple à l'aide des fonctions définies par les quadratures. Si l'on voulait donc se restreindre aux cas que l'on peut étudier avec des intégrales définies ou indéfinies, le champ de nos recherches serait singulièrement diminué, et l'immense majorité des questions qui se présentent dans les applications demeureraient insolubles.
Il est donc nécessaire d'étudier les fonctions définies par des équations différentielles en elles-mêmes et sans chercher à les ramener à des fonctions plus simples [...].
Rechercher quelles sont les propriétés des équations différentielles est donc une question du plus haut intérèt. On a déjà fait un premier pas dans cette voie en étudiant la fonction proposée dans le voisinage d'un des points du plan. Il s'agit aujourd'hui d'aller plus loin et d'étudier celte fonction dans toute l'étendue du plan. Dans cette recherche, notre point de départ sera évidemment ce que l'on sait déjà de la fonction étudiée dans une certaine région du plan. L'étude complète d'une fonction comprend deux parties:
1° Partie qualitative (pour ainsi dire), ou étude géométrique de la courbe définie par la fonction;
2° Partie quantitative, ou calcul numérique des valeurs de la fonction.» (Poincaré 1881, pp. 375-376).

His interest in the theory of differential equations accompanied much of his scientific output, from his first article in 1878 to his last in 1912. However, from 1885 onward, it is evident that his focus shifted increasingly toward celestial mechanics. Indeed, the articles he published from that year on concerning differential equations were mostly related to problems in celestial mechanics. In 1885, he published an article entitled "Sur l'équilibre d'une masse fluide animée d'un mouvement de rotation" in volume 7 of the journal «Acta Mathematica». In the same volume, on the first six pages, there appears a notice written by the publisher Gösta Mittag-Leffler, *Mittheilung, einen von König Oscar II gestifteten mathematischen Preis betreffend* ("Communication concerning a mathematical prize donated by King Oscar II," Mittag-Leffler 1885), announcing the prize sponsored by King Oscar II, for which Poincaré would compete and ultimately win with a submission on the first of the proposed topics:

«Étant donné un système d'un nombre quelconque de points matériels qui s'attirent mutuellement suivant la loi de NEWTON, on propose, sous la supposition qu'un choc de deux points n'ait jamais lieu, de représenter les coordonnées de chaque point sous forme de séries procédant suivant quelques fonctions connues du temps et qui convergent uniformément pour toute valeur réelle de la variable. Ce problème dont la solution étendra considérablement nos connaissances par rapport au système du monde, paraît pouvoir être résolu à l'aide des moyens analytiques que nous avons actuellement à notre disposition; on peut le supposer du mémoires, car LEJEUNE-DIRICHLET a communiqué peu de temps avant sa mort à un géomètre de ses amis qu'il avait découvert une méthode pour l'intégration des équations différentielles de la mécanique, et qu'en appliquant cette méthode il était parvenu à démontrer d'une manière absolument rigoureuse la stabilité de notre système planétaire. Malheureusement nous ne connaissons rien sur cette méthode, si ce n'est que la théorie des oscillations infiniment petites parait avoir servi de point de départ pour sa découverte. On peut pourtant supposer presque avec certitude que cette méthode était basée non point sur des calculs longs et compliqués, mais sur le développement d'une idée fondamentale et simple, qu'on peut avec raison espérer de retrouver par un travail persévérant et approfondi. Dans le cas pourtant où le problème proposé ne parviendrait pas à être résolu pour l'époque du concours, on pourrait décerner le prix pour un travail, dans lequel quelque autre problème de la mécanique serait traité de la manière indiquée et résolu complètement.»

This represented only the first of the four problems proposed in the competition, formulated by Karl Weierstrass (1815-1897), a member of the prize commission along with Charles Hermite (1822-1901). In fact, the question reflected Weierstrass's strong interest in the n-body problem. This topic is further explored in (Barrow-Green 1997), where we read in a footnote on page 70:

«In a letter dated 15 August 1878, Weierstrass told Kovalevskaya that he had constructed a formal series expansion for solutions to the problem but was unable to prove convergence, and in 1880/81 he gave a seminar on the problems of perturbation theory in astronomy. Despite Weierstrass' own difficulties with the problem, certain remarks made by Dirichlet in 1858 had led him to believe that a complete solution was possible, and hence his choice of the Problem as one of the competition questions. Weierstrass' interest in the problem is chronicled in (Mittag-Leffler 1912).» (Barrow-Green 1997, p. 70, footnote).

Weierstrass here refers to the Lindstedt series, discussed in the previous paragraph.

Poincaré won the prize in January 1889, although the result he presented did not fully meet the requirements of the question posed.[5]

In fact, he focused solely on the three-body problem and, instead of demonstrating that the Lindstedt series converges, his research led him to hypothesize – without being able to prove it – that they diverged. He was asked to prepare his memoir for prompt publication in *Acta Mathematica*. Thus, in volume 13 of the 1890 «Acta Mathematica», "Sur le problème des trois corps et les équations de la dynamique" was published. This article presented the main ideas of Poincaré and came to be considered the foundation of his later monumental work *Les méthodes nouvelles mécanique céleste*, which appeared in three volumes over the seven years from 1892 to 1899. It is in the *Mémoire* (Poincaré 1890) that we find the first original formulation of the so-called "Poincaré 1954 speech.[6]

The theorem, in its original formulation, is stated as follows:

> **Theorem 1** ***(Poincaré Recurrence)***. Supposons que le point P reste à distance finie, et que le volume $dx_1\ dx_2\ dx_3$ soit un invariant intégral[7]; si l'on considère une region r_0 quelconque, quelque petite que soit cette région, il y aura des trajectoires qui la traverseront une infinité de fois.[8]

The theorem, with its characteristic geometric nature, would become the forerunner of Birkhoff's studies and the birth of ergodic theory (Sinai 1976, Barrow-Green 1993, Chenciner 2012), as Kolmogorov asserted in his Amsterdam lecture:

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[5] See (Barrow-Green 1997), (Diacu and Holmes 1996), and (Dumas 2014) for full details.

[6] As noted by (Barrow-Green 1997, p. 113), the original formulation of the theorem already appears in the unpublished 1889 draft memoir *Sur le problème des trois corps et les équations de la dynamique, avec des notes par l'auteur – mémoire couronné du prix de S. M. le Roi Oscar II*. Although printed in 1889, it was never formally published.

[7] This means that the volume of the region is conserved.

[8] English translation: *Suppose that the point P remains at a finite distance, and that the volume $dx_1\ dx_2\ dx_3$ is an integral invariant; if we consider any region r_0, however small this region may be, there will be trajectories which will cross it an infinite number of times.*

«After the work of H. Poincaré, the fundamental role of topology for this range of problems became clear. On the other hand, the Poincaré-Carathéodory recurrence theorem initiated the "metrical" theory of dynamical systems in the sense of the study of properties of motions holding for "almost all" initial states of the system. This gave rise to the "ergodic theory", which was generalized in different ways and became an independent center of attraction and a point of interlacing for methods and problems of various most recent branches of mathematics (abstract measure theory, the theory of groups of linear operators in Hilbert and other infinite-dimensional spaces, the theory of random processes, etc.). At the preceding International Congress in 1950 the extensive paper by Kakutani was devoted to general problems of ergodic theory.» (Kolmogorov 1957, pp. 355-356).

This theorem, along with the theorem of non-existence of uniform integrals for the three-body problem and many other results developed in the memoirs, finds a more considered elaboration in the three volumes of *Les Méthodes nouvelles de la mécanique céleste* (Poincaré 1892-99).

The introduction to the first volume of the work is a remarkable historical document: Poincaré traces the state of the art in celestial mechanics and describes the developments to which he contributed in a clear and concise manner.

I quote here a few passages that are particularly significant for the historical reconstruction of the evolution of dynamics:

«Le Problème des trois corps a une telle importance pour l'Astronomie, et il est en même temps si difficile, que tous les efforts des géomètres ont été depuis longtemps dirigés de ce côté. Une intégration complète et rigoureuse étant manifestement impossible, c'est aux procédés d'approximation que l'on a dû faire appel. [...] Le but final de la Mécanique céleste est de résoudre cette grande question de savoir si la loi de Newton explique à elle seule tous les phénomènes astronomiques; le seul moyen d'y parvenir est de faire des observations aussi précises que possible et de les comparer ensuite aux résultats du calcul. Ce calcul ne peut être qu'approximatif et il ne servirait à rien, d'ailleurs, de calculer plus de décimales que les observations n'en peuvent faire connaître. Il est donc inutile de demander au calcul plus de précision qu'aux observations; mais on ne doit pas non plus lui en demander moins. Aussi l'approximation dont nous pouvons nous contenter aujourd'hui sera-t-elle insuffisante dans quelques siècles.
[...] Cette époque, où l'on sera obligé de renoncer aux méthodes anciennes, est sans doute encore très éloignée; mais le théoricien est obligé de la devancer, puisque son

œuvre doit précéder, et souvent d'un grand nombre d'années, celle du calculateur numérique.

[...] Ces méthodes, qui consistent à développer les coordonnées des astres suivant les puissances des masses, ont en effet un caractère commun oui s'oppose à leur emploi pour le calcul des éphémérides à longue

échéance. Les séries obtenues contiennent des termes dits séculaires, où le temps sort des signes sinus et cosinus, et il en résulte que leur convergence pourrait devenir douteuse si l'on donnait à ce temps *t* une grande valeur.

La présence de ces termes séculaires ne tient pas à la nature du problème, mais seulement à la méthode employée.

[...] Mais le savant qui a rendu à cette branche de l'Astronomie les services les plus éminents est sans contredit M. Gyldén[9]. Son œuvre touche à toutes les parties de la Mécanique céleste, et il utilise avec habileté toutes les ressources de l'Analyse moderne. M. Gyldén est parvenu à faire disparaître entièrement de ses développements tous les termes séculaires qui avaient tant gêné ses devanciers. D'autre part, M. Lindstedt a proposé une autre méthode beaucoup plus simple que celle de M. Gyldén, mais d'une portée moindre, puisqu'elle cesse d'être applicable quand on se trouve en présence de ces termes, que M. Gyldén appelle critiques.

[...] Il m'a semblé, d'autre part, que mes résultats me permettaient de réunir dans une sorte de synthèse la plupart des méthodes nouvelles récemment proposées, et c'est ce qui m'a déterminé à entreprendre le présent Ouvrage.»[10] (Poincaré 1892-99, vol. I, pp. 1-5).

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[9] Johan August Hugo Gyldén (1841-1896) was a Finnish astronomer primarily known for his work in celestial mechanics. Further details will be provided in §1.4.2.

[10] The Three-Body Problem is of such importance in astronomy, and is at the same time so difficult, that all efforts of geometers have long been directed toward it. A complete and rigorous integration being manifestly impossible, we must turn to the processes of approximation.

[...] The final goal of Celestial Mechanics is to resolve the great problem of determining if Newton's law alone explains all astronomical phenomena. The only means of deciding is to make the most precise observations, and then compare them to calculated results. This calculation can only be approximate, and it would be pointless to calculate to more decimals than observation can give us. It is therefore useless to ask more precision from calculation than from observation, but neither should we ask less. Furthermore, the approximation with which we can content ourselves today will be insufficient in several centuries.

[...] This era, when we will be obliged to relinquish old methods, is without doubt still quite distant. However, the theorist must anticipate it, because his work must precede, and often by a great number of years, that of the numerical calculator.

[...] These methods, which consist of developing the coordinates of the heavenly bodies in terms of the powers of the masses, have, in fact, a common character which is opposed to

The first volume dealt with periodic solutions and the non-existence of uniform integrals, as well as asymptotic solutions to the three-body problem, while the second volume focused on the multiple perturbation series methods developed up to that time by Newcomb, Gyldén, Lindstedt, and Bohlin, and their applications to the three-body problem.

Finally, the last volume, which appeared six years after the second, delved into integral invariants, periodic solutions of the second kind, and doubly asymptotic solutions, the latter introduced by Poincaré himself in the prize memoir.

The difficulties highlighted by the various methods listed by Poincaré regarding the convergence of power series are characteristic not only of problems in celestial mechanics, but of all problems "close" to integrable problems, with which perturbation theory deals.

For this reason, Poincaré defined, on page 32 of Volume 1, what he would call the *Problème général de la Dynamique*. Let us see what this entails, using the same nomenclature used by the French mathematician.

The general problem of dynamics. Let us consider the study of the motion of *q* material bodies, free to move in space; each of them is characterized by a mass $m_1, \ldots, m_q$, by the three spatial

their use for long-term calculation of the ephemerides.
The series obtained contain terms called secular, where time occurs outside the sine and cosine terms, with the result that their convergence would become doubtful if we were to give this time t a large value.
The presence of these secular terms is not basic to the nature of the problem, but only to the method used.
[...] However, the scientist who has given this branch of astronomy the most eminent service is without question Gylden. His work touches all parts of Celestial Mechanics, and it uses with ease all resources of modern analysis. Gylden has succeeded in eliminating entirely from his development all secular terms, which so troubled his predecessors.
On the other hand, Lindstedt has proposed a method much simpler than that of Gylden, but of less power, because it is no longer applicable when we are confronted with those terms which Gylden calls critical.
[...] On the other hand, it has appeared to me that my results permitted me to unite, in a sort of synthesis, the greater part of the new methods recently proposed, and this is what made me decide to undertake the present work. (*English translation by NASA, Dover Publications, New York, 1957*).

coordinates: (x_1, x_2, x_3) for the first body, (x_4, x_5, x_6) for the second body, . . . , $(x_{3q-2}, x_{3q-1}, x_{3q})$ for the last body and by the three spatial coordinates of the momentum (y_1, y_2, y_3) for the first body, (y_4, y_5, y_6) for the second body, . . . , $(y_{3q-2}, y_{3q-1}, y_{3q})$ for the last body, with respect to a fixed reference system.
Since Newton's formulation, we have seen that the equations of motion correspond to a system of n second-order differential equations.
With the Lagrangian and Hamiltonian formalisms the equations take a new form, becoming a system of first-order differential equations in a space of *2n* variables (double, with respect to the first), in which the coordinates of the points are identified by their positions and their momenta.[a]
A force acts on each body as a result of the gravitational interactions among the masses. This force is vectorial, with spatial components along the three directions: (F_1, F_2, F_3) for the first body, (F_4, F_5, F_6) for the second body, . . . , $(F_{3q-2}, F_{3q-1}, F_{3q})$ for the last body.
If the system is conservative, there exists a function V, called the *force function*[b] such that

$$T - V = const.$$

We can also define the live half force[c], which has the form:

$$T = \frac{y_1^2 + y_2^2 + y_3^2}{2m_1} + \frac{y_4^2 + y_5^2 + y_6^2}{2m_2} + \cdots + \frac{y_{3q-2}^2 + y_{3q-1}^2 + y_{3q}^2}{2m_q}$$

Thus, the equation of live forces can be written as:

$$T - V = const.$$

or more generally,

$$T - V = F(x_1, x_2, \ldots, x_{3q}, y_1, y_2, \ldots, y_{3q})$$

and the equations of motion are described by:

$$\frac{dx_i}{dt} = \frac{dF}{dy_i} \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i}$$

Now, proceeding analogously to formalize the three-body problem, Poincaré observed that, since two of the bodies have much smaller masses than the third, their masses can be written as the product of a small parameter μ and a finite value (e.g., $m_1 = \mu\alpha_1$ and $m_2 = \mu\alpha_2$, with α_1 , α_2 finite).

It may then be advantageous to develop F in powers of μ:

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \ldots$$

with F_0 independent of any variable y_i. Whatever the value of μ, F is a periodic function of period 2π with respect to the variables y_i.

Thus Poincaré defines the *general problem of dynamics* as the study of the canonical equations (4), under the assumption that the function F can be expanded in a power series as in (5) and supposing that the function F_0 depends only on the variables x_1, x_2, . . . and that the successive F_i being periodic of period 2π with respect to the variables y_i.

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[a] In modern terms, the space formed by the pairs (x, y), where x is the position vector and y is the momentum vector, constitutes a differential manifold known as *phase space*.

[b] Today it is referred to as *potential energy*.

[c] Today it is referred to as *kinetic energy*.

The general problem of dynamics represents the form taken by the new methods presented by Poincaré in his classic three-volume work.

Poincaré's own description of his contributions, in his "Analyse des travaux scientifiques de Henri Poincaré faite par lui-même", published in 1921 in volume 38 of «Acta Mathematica» (Poincaré 1921), deserves close attention. In the 133 pages, all his publications are first listed and then, in what he called the "Résumé analytique," seven research areas are considered:

«J'ai classé les travaux que j'ai à résumer sous les sept rubriques suivantes:
1°. Equations Différentielles.
2°. Théorie générale des Fonctions.
3°. Questions diverses de Mathématiques pures (Algèbre, Arithmétique, Théorie des Groupes, Analysis Situs).
4°. Mécanique Céleste.
5°. Physique Mathématique.
6°. Philosophie des Sciences.
7°. Enseignement, vulgarisation, divers (Bibliographie, rapports divers).» (Poincaré 1921, p. 36).

The section on celestial mechanics is divided into very short and discursive subsections. The first is entitled "Généralités sur les Équations de la Dynamique et de la Mécanique Céleste":

«Les équations de la Dynamique présentent des propriétés remarquables qui ont été mises en évidence par JACOBI dans ses Vorlesungen[11].
Quelles sont les conséquences plus ou moins immédiates de ces propriétés? Quel partie peut-on en tirer pour la mise en équation des problèmes de Dynamique et en particulier des problèmes de Mécanique Céleste? Telle est la première question dont je veux parler ici.
J'ai été amené à passer en revue les principales propriétés des équations canoniques. Les propriétés sont classiques; et je n'ai eu qu'a perfetionner certains détails; en me servant surtout du caractère bien connu qui permet de reconnaître si un changement de variables conserve la forme canonique des équations.
Ce genre de transformations facilite la mise en équation du problème des trois corps; c'est ce que j'ai montré. On sait que dans le procédé classique on rapporte toutes les planètes à des axes mobiles passant par le Soleil. L'inconvénient est que la fonction perturbatrice n'est pas la même pour toutes les planètes. Un autre procédé consiste à rapporter chaque planète au centre de gravité du système formé par le Soleil et toutes les planètes inférieures à celle que l'on considère. L'inconvénient est évité, mais la fonction perturbatrice est un peu plus compliquée. J'ai proposé un troisième procédé, dans lequel les coordonnées de chaque planète sont rap- portées au Soleil, et sa vitesse à des axes fixes.

[11] *Vorlesungen über Dynamik (Lectures on Dynamics)* by Karl Gustav Jakob Jacobi (1804-1851), first published in 1866.

Malgré les travaux dont les équations canoniques ont été l'objet depuis JACOBI, toutes leurs propriétés ne sont pas connues, ou plutôt on n'a pas insisté sur toutes les formes que peuvent revêtir ces propriétés et qu'il peut être utile de connaître. Si par exemple on étudie les équations aux variations des équations de la Dynamique, c'est à dire les équations qui définissent une solution infiniment peu différente d'une solution donnée, on rencontre des propositions importantes sur lesquelles j'ai attiré l'attention.
D'un autre côté, j'ai été amené à introduire une notion nouvelle, celle des invariants intégraux. Ce sont certaines intégrales définies simples ou multiples qui demeurent constantes, quand le champ d'intégration varie conformément à une certaine loi définie par une équation différentielle. Si par exemple on envisage les équations différentielles au mouvement d'un fluide incompressible, le volume est un invariant intégral.
Les équations canoniques de la Dynamique possèdent des invariants intégraux remarquables et l'existence de ces invariants jette une grande lumière sur leurs propriétés.
Pour en finir avec ces généralités sur les équations de la Dynamique et le problème des 3 corps, je signalerai un dernier travail. On sait que BRUNS a démontré que le problème des 3 corps ne saurait admettre d'autre intégrale algébrique que les intégrales classiques. Malheureusement dans sa démonstration subsistait une lacune grave et particulièrement délicate à combler. J'ai été assez heureux pour mettre la belle et ingénieuse démonstration de M. BRUNS à l'abri de toute objection.» (Poincaré 1921, p. 102).

The proof to which he refers in the last sentence is found in Chapter 5 of the first volume of Poincaré (1892-99), on page 233. It is now known as "Non-existence des intégrales uniformes" (Fermi 1923b; Benettin *et al.* 1985).

To this theorem must be added the discovery of certain complex solutions – called by Poincaré *asymptotic* and *doubly asymptotic* (i.e., in infinite past and future time) – as well as of trajectories, called *homoclinic*, so intricate that they cannot be drawn. This marked the beginning of a very delicate moment in the history of celestial mechanics: once it became clear that the method of direct integration could no longer be pursued and that some were anything but simple, the study of the qualitative and global aspects of motion became the new paradigm.

Moreover, the possibility that the mathematical description of the system might be inherently unstable began to gain ground, gradually creating an ever-widening gap between astronomers, with their direct measurements, and mathematicians, with their increasingly chaotic theories of the Solar System.

1.1.2 Ferrying classical mechanics into the 20th Century: Edmund Whittaker's (1873-1956) *A treatise on the analytical dynamics of particles and rigid bodies* (1904)

In the late 19th century, celestial mechanics and the problem of the stability of the Solar System were at the center of interest for the international mathematical community, thanks in part to Poincaré's revolutionary new methods. The question posed by the French mathematician himself – *de savoir si la loi de Newton explique à elle seule tous les phénomènes astronomiques* – remained open. Although Poincaré's work was widely recognized during his lifetime, many aspects of it remained enigmatic afterwar[12]. This was due mainly to two factors: on the one hand, Poincaré did not make an effort to condense, refine, or carefully verify his work (Dumas 2014); on the other hand, mathematics in the 20th century underwent a restructuring of its conceptual framework. Previous theories were not discarded, but rather absorbed into new paradigms, rendering them almost unrecognizable from the perspectives of the original authors. The field of classical mechanics was not spared from this restructuring. On this subject, Severino Collier Coutinho remarks in a 2014 paper:

«So dramatic have been the changes that mechanics has undergone in the twentieth century that the style and even the contents of most books on dynamics written before the 1930s look hopelessly dated to present-day readers. But there are exceptions.» (Coutinho 2014, p. 356).

Regarding the "exceptions", Coutinho refers to the work of the British mathematician of Scottish origin, born in 1873, when Poincaré was just 18 years old: Edmund Taylor Whittaker (1873-1956).

In the year of publication of the last of the three volumes of *Les*

[12] (Dumas 2014, p. 43).

méthodes nouvelles, Whittaker was in Cambridge, England, when the British Association asked him to prepare a report on the state of research on the three-body problem. The English mathematician and astronomer William Hunter McCrea (1904-1999) reported the notice in (McCrea 1957):

«Whittaker's interests in dynamics and optics were closely linked with an interest in their astronomical applications. As early as 1898 the Council of the British Association resolved "that Mr. E. T. Whittaker be requested to draw up a report on the planetary theory". Besides, in those days an interest in astronomy was more general amongst mathematicians than it has since become, and most professional mathematicians in the country joined the Royal Astronomical Society.» (McCrea 1957, p. 236).

The following year, Whittaker wrote *Report on the Progress of the Solution of the Problem of Three Bodies*, a report covering the last thirty years of research, up to and including Poincaré's very recent studies:

«The Report attempts to trace the development of the subject in the last thirty years, 1868-98; this period opens with the time when the last volume of Delaunay's "Lunar Theory" was newly published; it closes with the issue of the last volume of Poincaré's "New Methods in Celestial Mechanics". Between the two books lies the development of the new dynamical astronomy.
The work will be distributed under the following seven headings:
§I. The differential equations of the problem.
§II. Certain particular solutions of simple character.
§III. Memoirs of 1868-89 on general and particular solutions of the differential equations, and their expression by means of infinite series (excluding Gyldén's theory).
§IV. Memoirs of 1868-89 on the absence of terms of certain classes from the infinite series which represent the solution.
§ V. Gyldén's theory of absolute orbits.
§ VI. Progress in 1890-98 of the theories of §§ III and IV
§ VII. The impossibility of certain kinds of integrals.» (Whittaker 1899, p. 122).

Section VI is dedicated to the developments by Poincaré in *Les Méthodes nouvelles de la mécanique céleste*. It provides a very accurate description of some of the aspects addressed by the French mathematician (such as periodic and asymptotic solutions and invariant integrals); he highlights the importance of certain results, such as the recurrence theorem (Whittaker 1899, p. 145), and reports the original formulation of the "fundamental problem of dynamics" (Whittaker 1899, p. 147).

Whittaker held the position of Secretary of the Royal Astronomical Society from 1901 to 1906, became Astronomer Royal of Ireland, moved to Dunsink Observatory – the same observatory where Hamilton had worked – and was appointed Professor of Astronomy at Dublin University in 1906. Five years after his report on the problem of three bodies, he published the first edition of his monumental work on analytical mechanics, entitled *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies; with an Introduction to the Problem of Three Bodies*.

McCrea emphasizes the importance of Whittaker's figure in British mathematics, especially with reference to his work in the field of dynamics:

«The name of Sir Edmund Whittaker will always hold a unique place in the history of British mathematics. It may reasonably be claimed that no single individual in this century or the last had so far-reaching an influence upon its progress. If such a claim comes as a surprise to some present-day readers, it is probably because we are apt to forget the part that Whittaker played personally in bringing about so many of the developments that we now take for granted.

British nineteenth-century mathematics was deplorably insular, apart from the work of a very few of its most distinguished men in certain particular fields. Whittaker, more than anyone else, brought about the transformation to something that was more abreast of developments elsewhere while, happily, still bearing characteristic features of its own.

[...] He was the first to make available in this country a comprehensive account of the special functions of analysis. Further, what Forsyth[13] and Whittaker did for analysis, Whittaker alone did for applied mathematics by his Analytical Dynamics.

[...] Moreover, with an inspired appreciation of what is in the best sense useful in mathematics, he has included in his books much that was found to be needed in the development of quantum mechanics and wave-mechanics more than twenty years afterwards. The part that British workers in particular were thus enabled to contribute

[13] Andrew Russell Forsyth (1858-1942) was a British mathematician, and Whittaker was his only (at least officially recorded) student. Forsyth authored significant works on analysis that played a key role in introducing foreign mathematical research to Britain.

to this development owes a debt to Whittaker which seems scarcely to have been sufficiently acknowledged.» (McCrea 1957, p. 234).

Whittaker's *Analytical Mechanics* was the first book to provide a systematic account in English of the theory arising from Hamilton's equations (McCrea 1957). In a recent paper, Coutinho traced the history and wide spatial and temporal diffusion of the essay. He attempted to study the reasons why this work has remained so enduring, even at times when many contemporary works were shelved and deemed obsolete. Published in 1904, it went through four editions, was translated into German and Russian, and is still in print today:[14]

«What were the qualities that allowed Whittaker to write a book [...] that remains useful to mathematicians working in several different areas, more than one hundred years after it was written? [...] this was in good measure due to Whittaker's great knowledge of the literature and to his ability to organize this knowledge in a systematic way. Moreover, his reading was not limited to contemporaneous works, it also encompassed the classics of the 18th and 19th centuries. [...] It seems to me that the success of Whittaker 's books owes much to the fact that he was one of that rare breed, a scientist who is also a scholar, of which D'Arcy Thompson is probably the best known representative. People whose research may not have been exceptional, but whose great knowledge of the literature, including historical works, allowed them to "crystallize" in their books a vision of a whole subject that would greatly influence later generations.» (Coutinho 2014, p. 403).

1.1.3 The Scandinavian research tradition: *Die Mechanik des Himmels* (1902-07) by Carl Ludvig Charlier (1862-1934)

For almost two centuries, until 1809, the Finnish-Swedish union was not only geographical and political, but also manifested itself in the links between the universities and academies of the two countries. Astronomy and celestial mechanics represented privileged fields of study and, from the second half of the eighteenth century, with the construction of the Stockholm,

[14] An edition dated 27 December 2022 by Cambridge University Press is currently available for sale.

Uppsala, and Lund observatories, research intensified.[15] Given their geographical position, the two countries could count on collaborations with Germany to the west and, subsequently, with Russia to the east. Furthermore, after Sweden's defeat against Russia in 1809 and the cession of Finland to Russia – becoming the Grand Duchy of Finland – new opportunities opened up for Finnish astronomers and celestial mechanicians to collaborate with their Russian neighbours. In fact, while in the eighteenth century most of the studies were under Swedish dominion, the situation of astronomical research changed along with the shifting political balance. Between 1831 and 1834, on the Ulrikasborg Hill (Observatory Hill Park) in Helsinki, the architect Carl Ludvig Engel (1778-1840), in collaboration with Professor Friedrich Wilhelm Argelander (1799-1875), completed the construction of one of the most modern observatories of that period. The Helsinki Observatory ended up influencing the next major observatory project in the Russian Empire, namely the main imperial observatory at nearby Pulkovo, just south of St. Petersburg.

The lively community of Finnish astronomers, supported by their observatory and newly formed connections, enjoyed great opportunities to receive an excellent education at Pulkovo – more so than they would have under continued Swedish rule. Representative of this scientific fervour was Karl Frithiof Sundman (1873-1949), a Finnish mathematician and astronomer who, after graduating in 1897, went to the Pulkovo Observatory to continue his research in astronomy. He demonstrated the existence of a solution in convergent infinite series to the three-body problem, using analytical methods for the regularization of motion – i.e., the elimination of singularities through a suitable series of transformations.

On the other hand, from neighbouring Sweden, the main figures were Anders Lindstedt, Johan August Hugo Gyldén – names already encountered in the previous paragraphs – and Carl Ludwig Charlier.

Charlier (1862-1934) defended his thesis *Untersuchung über die allgemeinen Jupiter-Störungen des Planeten Thetis*[16] in 1887, as a student of Gyldén, at Uppsala University. Thanks to the quality of his work, he was immediately appointed professor at the same university.[17] In the autumn of 1898, he gave lectures on general celestial mechanics, which contained – as he himself revealed in

[15] (Holmberg 1999).

[16] Investigation of the general perturbations by Jupiter on the planet Thetis.

[17] (Wicksell 1935).

the preface of the first volume (Charlier 1902-07) – the main topics of his two volumes *Die Mechanik des Himmels*, published in 1902 and 1907.

In the preface on page iii, he declares the main intent of the texts:

«Als Ziel habe ich mir gesteckt, eine möglichst einheitticlie Darstellung des jetzigen Standpunkts der Untersuch nugen über die Mechanik des Himmels, insofern sieb dieselbe mit der Bewegung von Massenpunkteu beschäftigt, zu geben. Es ist dabei mein Hauptstreben gewesen, die astronomisch wichtigen Resultate hervorzuheben, indem ich gleichzeitig die mathematische Eleganz und Schärfe, welche besonders die neueren Untersuchungen auf diesem Gebiete[18]». (Charlier 1902-7, vol. 1, p. III).

Like and contemporary with Whittaker's works, the volumes represent a systematization of motion – that is, the elimination of singularities through a suitable series of transformations.[19]

And it is Whittaker who cites the results of the Swedish mathematician several times, already in his 1899 report for the British Association:

«Poincaré's paper gave a fresh stimulus to the investigation of periodic solutions. In 1890 v. Haerdtl[20] calculated numerically two cases of the restricted problem of three bodies. Charlier in 1892 discussed the same cases by means of expansions proceeding in ascending powers of the time, and the same author in 1893 found a set of periodic solutions of the problem of three bodies in a plane, whose expansion involves four arbitrary constants. [...] Brown[21] in 1897 discussed the properties of the general solution in trigonometric series of the problem of three bodies, by supposing it to have been derived by integrating the Hamilton-Jacobi equation.
[...] Researches relating to the convergence of the trigonometric series of dynamical astronomy were published in 1896 by Charlier and in 1898 by Poincaré. The former,

[18] English translation: «My goal has been to provide as uniform a presentation as possible of the current point of view in investigations of celestial mechanics, regarding the motion of mass points. My main objective has been to emphasize the astronomically significant results, while also aiming to convey the mathematical elegance and precision made possible particularly by the most recent research in this field».

[19] (Sundman 1907, 1910, 1913).

[20] Eduard Freiherr von Haerdtl (1861-1897) was an Austrian astronomer, who became the first professor of astronomy at the University of Innsbruck in 1892.

[21] Ernest William Brown (1866-1938) was an English mathematician and astronomer, known in the field of celestial mechanics for his studies on lunar movements.

by expanding in descending powers of *m* the coefficient of the *m*th term in such a series, arrived at the conclusion that the convergence can be augmented by dividing the function expressed into two parts, one of which depends on the first terms in these expansions of the coefficients.» (Whittaker 1899, pp. 151, 156-157).

The question concerning the singularities of motion in the three-body problem also found fertile ground in the Scandinavian scientific environment, which played a leading role in the history and evolution of astronomy and celestial mechanics during those years.

In the last decade of the 19th century, the theory of singularities in the three-body problem was developed by the French mathematician Paul Prudent Painlevé (1863-1933).

Singularities are closely related to collisions between bodies, since each collision corresponds to a singularity in the differential equations of the problem. Therefore, the goal was to eliminate singularities in order to study the motion of the system even after a possible collision. Furthermore, the question arose as to whether singularities result only from collisions or whether other phenomena might also give rise to them.

In 1896, Painlevé published "Sur les singularités des équations de la Dynamique" (Painlevé 1896), an in-depth study in which he demonstrated that the only possible singularities were those due to collisions.

1.1.4 Classical and modern mechanics: Jean-François Chazy (1882-1955) and the capture in the Three-Body problem

The 20th century ushered in the advent of the theories of relativity and quantum mechanics, and celestial mechanics – like other fields of classical mechanics – was significantly affected by the ongoing restructuring of the sciences. Classical mechanics, a foundational pillar for 19th-century mathematicians and a source of modern European mathematical development, became increasingly marginalized – though not without some notable exceptions.

Upon closer examination, one can identify two fundamental factors contributing to the transformation that occurred at the beginning of the last century. On the one hand, in the words of Dumas (Dumas 2014, p. 7), "not surprisingly, in that period, physicists abandoned classical mechanics to the few hardy mathematicians who remained interested in it. The physicists returned with wondrous stories of their exploits in quantum mechanics, relativity, and nuclear physics."

On the other hand, it was precisely the formulation of the principles of the theory of relativity that cast doubt on the validity of Newton's laws and Galilean transformations – transformations that relate the coordinates describing the same phenomenon from two distinct reference frames. Classical theories began to appear obsolete, and celestial mechanics perhaps needed to be revised in light of the new relativistic ideas.

Although there was widespread belief in a clear break between classical and modern theories – or in the outright replacement of the older paradigm by the new one – what actually occurred among the few mathematicians who continued to study celestial mechanics was a coexistence of the two. There are numerous examples of scholars who engaged with both celestial mechanics and relativity.

Poincaré himself addressed questions concerning the simultaneity of time and Lorentz transformations – which replaced the Galilean ones – in Poincaré (1900), even before Einstein formulated the theory of relativity in 1905. The Italian mathematician Tullio Levi-Civita (1873-1941) and the French mathematician and astronomer Jean-François Chazy (1882-1955) are among those who made contributions in both fields.

After being mobilized into the French army in 1914 and assigned to the sound reconnaissance laboratory established at the École Normale Supérieure in Paris, Chazy did not return to his research at the University of Lille until 1919. He published extensively on the three-body problem – for example, (Chazy 1922, 1924, 1929) – and also worked on the subject of relativity and its application to celestial mechanics. He published the essay *La théorie de la relativité et la mécanique céleste* (Chazy 1928-1930) in two volumes:

«On trouve, dans les deux livres de Jean Chazy, toutes les notions nécessaires de géométrie différentielle générale, les méthodes générales de calculs et de formation des équations d'Einstein, l'étude des questions classiques.
Nous insisterons sur le problème du périhélie de Mercure qui avait été l'objet, nous l'avons vu, d'une discussion approfondie par Jean Chazy.» (Darmois 1957, pp. 42-43).[22]

[22] Georges Darmois (1888-1960) refers in particular to certain applications of relativity to the motion of Mercury – the only planet in the solar system for which, due to its proximity

Chazy's main contribution to the three-body problem concerned the final behavior of the system's motion – in other words, considering the values of the time variable as it approaches infinity. He classified seven possible final trajectories: hyperbolic motions, hyperbolic-elliptic motions, oscillating motions, constrained motions, parabolic-elliptic motions, hyperbolic-parabolic motions, and parabolic motions. Chazy analyzed each of these in detail.

In particular, he theorized the impossibility of capture in the three-body problem. The French mathematician Darmois, in his *Notice sur la vie et les travaux de Jean Chazy (1882-1955)*, published in 1957, wrote on this topic:

«Les résultats ainsi obtenus, qui assujettissent le point représentatif à demeurer dans une région ou sur une surface, ont permis à Jean Chazy d'affirmer l'impossibilité dans certains cas de l'écartement indéfini correspondant à une dislocation d'un système. C'est ainsi que si l'un des corps vient de l'infini (dans une direction non parallèle au plan du mouvement des deux autres), il ne peut que s'en éloigner indéfiniment au bout d'un temps fini passé en leur voisinage. Les deux corps reviennent alors à un mouvement relatif elliptique.
Ce résultat généralisait et précisait une étude de Schwarzschild[23] faite dans le cas d'un troisième corps de masse nulle. Signalons que de nouvelles recherches sont entreprises, surtout en URSS, sur ce sujet.» (Darmois 1957, p. 40).

We will see, in fact, that the developments in the USSR to which he referred had already been published in 1947 and 1953 by the scientists Kirill Aleksandrovich Sitnikov (b. 1926) and Otto Yulyevich Schmidt (1891-1956) (Schmidt 1947; Sitnikov 1953). Their work provided counterexamples to the validity of Chazy's capture theory in the three-body problem, effectively refuting the conclusions of the French mathematician.

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to the Sun, the theory of relativity has produced more precise results than classical theories. These applications revealed the presence of a still unexplained body, despite unsuccessful efforts to identify perturbing masses.

[23] Karl Schwarzschild (1873-1916), a German mathematician, astronomer, and astrophysicist.

1.1.5 Otto Yulyevich Schmidt (1891-1956): A soviet contribution in 1947

> «Pour une époque comme le premier tiers du XX^e siècle, il est en général difficile d'étudier la science astronomique à l'intérieur des frontières d'un pays. En effet, dès cette époque, l'astronomie est une science internationale du point de vue de la collaboration et de la coordination des recherches. Cette collaboration et cette coordination ont d'ailleurs été sensiblement renforcées après la constitution de l'Union Astronomique Internationale, en 1919.
> L'étude de l'astronomie en U.R.S.S. de 1917 à 1935 a attiré notre attention, car ce pays constituait une exception à cette règle.»
>
> (Nicolaïdis 1984, p. 6).

The Soviet mathematician, astronomer, and explorer Otto Yulyevich Schmidt (1891-1956) was cited by Kolmogorov as one of the sources of inspiration for his contributions to mechanics.[24] Specifically, Kolmogorov included in the bibliography of the published text of his Amsterdam lecture a paper by Schmidt entitled "On Possible Capture in Celestial Mechanics", published in the «Doklady Akademii Nauk SSSR» in 1947 (Schmidt 1947).

Born in Mogilev (now Belarus), Schmidt held a brief post as professor of mathematics at the University of Kiev in 1915, having graduated from the same university in 1913. In 1923, he became a professor of mathematics at Moscow University, and in 1929, he was appointed head of the algebra department, where he founded an active school of group theory.

His professional life was divided between academic and administrative roles. He held various institutional positions, including serving as head of one of the divisions of the People's Commissariat for Food, created in 1917 following the dissolution of the Ministry of Food by the Bolsheviks. In April 1924, he was appointed editor-in-chief of the *Great Soviet Encyclopedia* – a project undertaken in three editions launched in 1926, 1949, and 1977. As Laurent Mazliak describes it, it was "a gigantic enterprise to the glory of 'Marxist science' and of the Soviet regime" (Mazliak 2018, p. 25). Schmidt remained chief editor until 1941.

Kolmogorov was a principal contributor to the Encyclopedia, publishing more than one hundred entries between 1937 and 1975. He was tasked

[24] In his conversation with Arnol'd, which I analize in Chapter 2, §2.1.

with writing the entry "Mathematics" for all three editions, published in 1938, 1950, and 1974, respectively.[25]

Their collaboration on the *Encyclopedia*, along with their positions as colleagues at the same university, undoubtedly enabled the two mathematicians to engage with one another and exchange ideas. Several comments attest to this, including one written by Kolmogorov himself regarding his interest in the theory of turbulence:

«In 1946 O. Yu. Shmidt suggested that I should head the Turbulence Laboratory in the Institute of Theoretical Geophysics, USSR Academy of Sciences. In 1949 this post was passed to Obukhov. I was not engaged in experimentation myself, but I worked extensively with other researchers on computation and graphical processing of the data.» Kolmogorov's words in (Tikhomirov 1991, p. 902).

Schmidt was a mathematician, but also an explorer and astronomer, although his research in the latter field dates only to the final decade of his life. In 1949, he published the book *A Theory of Earth's Origin* (in Russian), which compiled the content of four lectures delivered at the Geophysical Institute of the USSR Academy of Sciences in 1948:[26]

«The problem of the origin of the Earth is one of such great importance to science that it possesses interest not only for the specialists – astronomers, geophysicists, geologists, geographers and others – but also for the general public. The Soviet people have made very considerable cultural progress so that it is only natural that they should show an interest in this problem and demand an answer from their scientists: the problem of the Earth's origin, say our people, must be solved as quickly as possible on account of its specific importance to the study of nature and from the standpoint of our philosophy of dialectical materialism.
The author's hypothesis of the genesis of the Earth and other planets proposed in 1944 met with a wide response, gave rise to extensive criticism and discussion. In the course of time the hypothesis has developed and grown into a detailed theory.
Apart from separate publications in scientific journals it became necessary to publish, at least, an interim report on basic results and methods: The First Edition of this little booklet was published in 1949: it consisted of four lectures which I delivered

[25] See also (Graham 1993) and (Mazliak 2018) for further references.

[26] Author's preface to the second edition, English version in (Schimdt 1958, p. 7).

at the Academy of Sciences Geophysical Institute in 1948.»

One of the articles referred to by Schmidt is precisely (Schmidt 1947) – the same work cited by Kolmogorov.

The reason for this renewed interest in celestial mechanics and astronomy in the late 1940s may also lie in Schmidt's prudence in addressing such a politically sensitive topic during the 1930s – a period that, as we will see in the next chapter, led to the dramatic vicissitudes of the purge of astronomers in 1936-1937. His caution has also been noted by Mazliak, who observes that, unlike other contributors to the *Great Soviet Encyclopedia*, Schmidt was spared from the political persecutions of that era:

«[…] a large majority of its [first editorial board] members were victims of the political storms experienced by Soviet Union in the 1930s. It is therefore slightly surprising that Otto Schmidt could remain at the head of the enterprise almost until the end (he resigned in fact in 1941), despite his proximity with Bukharin and even, to a certain extent, with Trotsky. Maybe Stalin thought it was useless for the regime to touch an internationally too well-known scientist. But above all, Schmidt himself had the wisdom, as soon as the end of the 1920s, not only to make a brilliant come back to mathematics (he was appointed to the newly created Chair of higher algebra at Moscow university in 1929 and remained there until 1949), but also to participate to long-distance scientific exploratory expeditions such as the German-Soviet expedition to the Pamir (1928) and afterwards the long expedition in the Arctic (1930-1934), which maintained him far from the internal struggles tearing the party apart at the turn of the 1930s.» (Mazliak 2018, p 35).

And, far removed from the years of the purges, Schmidt's interest in topics related to celestial mechanics is evident. It is enough to note that in the introduction to (Schmidt 1958), he emphasizes the specific importance of celestial mechanics to the study of nature and to the philosophical perspective of dialectical materialism.

In a 1972 article by the Japanese astronomer Yusuke Hagihara, titled "Recent Advances of Celestial Mechanics in the Soviet Union" (Hagihara 1972), the first section is dedicated precisely to the problem of capture, beginning with Schmidt's contributions.

In §3.3, I will explore the significance of Schmidt's work in celestial mechanics within the context of Kolmogorov's research program.

1.2 Metrical and spectral studies: modern ergodic theory and the theory of dynamical systems in the 1930s

The essays by Whittaker and Charlier helped the international scholarly community recognize that the ancient and illustrious discipline of mechanics required new horizons for theoretical development. What, then, were the subsequent developments in the first decades of the 20th century? In this paragraph, I consider a series of contributions – many inspired by Poincaré's approach – that Andrej Kolmogorov regarded as crucial to the development of his work on classical mechanics. In particular, he referred to the evolution of the theory of dynamical systems at the beginning of the century, the emergence of ergodic theory, and the contributions to nonlinear mechanics made in the Soviet Union during the 1930s.

Historians of science have largely shifted their attention away from the evolution of classical mechanics after 1900, focusing instead on the nascent physical theories of the twentieth century: the theory of relativity and the new quantum mechanics. In his 1957 *Essay on the History of Mechanics* (Dugas 1957), René Dugas traces the origins of dynamics to the end of the Middle Ages, its rapid development through the works of Kepler, Galileo, Descartes, Huygens, and Newton, and its refinement by Euler, Lagrange, Laplace, and Hamilton. This lineage culminates in the reflections of the French scholars Poincaré, Paul Prudent Painlevé (1863-1933), and Pierre Maurice Marie Duhem (1861-1916). Dugas then turns to what he describes as the "modern physical theories of mechanics." Around this new paradigm, a distinct scientific community emerged: the so-called theoretical physicists. The rise of these modern theories led to the retroactive labeling of earlier approaches as "classical", referring to studies in mechanics grounded in the 19th-century mathematical tradition.

While modern mechanics attracted broad attention – including from the general public – and relegated classical approaches to the background, research in the wake of Poincaré continued, particularly in the United States and the Russian Empire. These were two relatively young mathematical communities, peripheral within the international scientific landscape. It was a time marked by war and political totalitarianism, yet the connection between the two countries remained strong, encompassing fields such as education, agricultural planning, astronomy, and mathematics. Poincaré's "new methods" in celestial mechanics served as a key source of ideas and inspiration during this transitional period. It was a time of profound change, during which the theory of dynamical systems was conceived – a development that aimed to extend the scope of differential equations to encompass time-evolving phenomena beyond the motion of inanimate bodies. As in earlier periods, research in mechanics was closely linked to research in mathematical analysis, which experienced significant advancements during those years.

In 1985, the first of three volumes of *Selected Works by Kolmogorov*, titled *Mathematics and Mechanics*, was published in Russian. The volume was edited by the Russian mathematical physicist Vladimir Mikhailovich Tikhomirov (b. 1934), a student of Kolmogorov. In it, Kolmogorov includes commentary on his three major works on mechanics from the 1950s:

«My papers on classical mechanics appeared under the influence of von Neumann's papers on the spectral theory of dynamical systems and, particularly under the influence of the Bogolyubov-Krylov paper of 1937. I became extremely interested in the question of what ergodic sets (in the sense of Bogolyubov-Krylov) can exist in the dynamical systems of classical mechanics and which of the types of these sets can be of positive measure at present this question still remains open). To accumulate specific information we organized a seminar on the study of individual examples. My ideas concerning this topic and closely related problems aroused wide response among young mathematicians in Moscow.» (Kolmogorov 1991, p. 521)[27].

The Hungarian scholar John von Neumann (1903-1957), a rising star in German mathematics between the two world wars, was active in both classical and quantum mechanics. In fact, his contribution to classical mechanics appears to have been encouraged by his contact with his senior colleague George Birkhoff (1884-1944), a relationship established in the late 1920s when von Neumann began his visits to the United States. As we shall see, a young American collaborator of Birkhoff's, Bernard Koopman (1900-1981), served as a bridge between Birkhoff and von Neumann.

As early as 1911, Birkhoff had begun his work in classical mechanics in the wake of Poincaré, laying the foundations for the theory of dynamical systems. In the interwar years, he emerged as a leading figure not only in the United States but also among scholars working in the Soviet Union – thus

[27] The English text presented here was published in the English edition, translated by the Russian mathematical physicist Vladimir Markovich Volosov, and released in 1991.

contributing to the international recognition of the then-young American mathematical community.

In this paragraph, I examine the contributions of Birkhoff, von Neumann, and Koopman in the early 20th century in the United States and trace a line that connects their work to that of the Soviet scholars Nikolay Mitrofanovich Krylov and Nikolay Nikolaevich Bogoliubov, who collaborated in the 1930s in Kiev, at the Ukrainian Academy of Sciences, on the study of linear and nonlinear mechanical oscillations.

1.2.1 General dynamical systems: George David Birkhoff's work in the wake of Poincaré (1912-1927)

> «In a paper recently published in the «Rendiconti del Circolo Matematico di Palermo» (vol. 33, 1912, pp. 375-407) and entitled "Sur un théorème de Géométrie", Poincaré enunciated a theorem of great importance, in particular for the restricted problem of three bodies; but, having only succeeded in treating a variety of special cases after long-continued efforts, he gave out the theorem for the consideration of other mathematicians.
> For some years I have been considering questions of a character similar to that presented by the theorem and it now turns out that methods which I have been using are here applicable. In the present paper I give the brief proof which I have obtained, but do not take up other results to which I have been led.»
>
> (Birkhoff 1913, p. 14).

When publishing the above-quoted words, George David Birkhoff (1884-1944) was 28 years old and had just been appointed assistant professor at Harvard. The theorem to which Birkhoff refers is stated as follows:

Theorem 2. ***Poincaré's geometric theorem***.[28] Let us suppose that a continuous one-to-one transformation T takes the ring R, formed by concentric circles C_a and C_b of radii a and b respectively ($a > b > 0$), into itself in such a way as to advance the points of C_a in a positive sense, and the points of C_b in the neg-

[28] This statement is taken from (Birkhoff 1912, p. 14).

ative sense, and at the same time to preserve areas. Then there are at least two invariant points.[29]

As Florin Diacu and Philip Holmes recall in their essay *Celestial Encounters: The Origins of Chaos and Stability* (1996), Poincaré once wrote to the editor of the Palermo journal «Rendiconti del Circolo Matematico» as if anticipating his own imminent death:

«At my age, I may not be able to solve it, and the results obtained, which may put researchers on a new and unexpected path, seem to me too full of promise, in spite of the deceptions they have caused me, that I should resign myself to sacrificing them.» (Poincaré in 1912, quoted in Diacu, Holmes 1996, p. 53).

Poincaré died on July 17, 1912, and three months later Birkhoff submitted a paper – quoted in part above – to the «Transactions of the American Mathematical Society», presenting a proof of the theorem. The article was published in January 1913. A testament to Birkhoff's deep intellectual connection to Poincaré was written by Marston Morse[30] (1892-1977) in his note "George David Birkhoff and His Mathematical Work":

«Birkhoff admired Moore of Chicago, but not to the point of imitating him. He respected Bôcher[31] no less, and did him the honor next to Poincaré of following his mathematical interests. F. R. Moulton's study of the work of Poincaré had something to do with Birkhoff's own intense reading of Poincaré. Poincaré was Birkhoff's true teacher. There is probably no mathematician alive who has explored the works of Poincaré in full unless it be Hadamard, but in the domains of analysis Birkhoff wholeheartedly took over the techniques and problems of Poincaré and carried on.» (Morse 1946, p. 357).

Poincaré's influence is evident in several of Birkhoff's publications

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[29] By *invariant point,* we mean a point of the ring that remains fixed under the transformation T.

[30] Harold Calvin Marston Morse (1892-1977) was an American mathematician, known for developing variational theory in general with applications to equilibrium problems in mathematical physics, a theory which is now called Morse theory.

[31] Maxime Bôcher (1867–1918) was an American mathematician who worked on differential equations, series, and algebra.

from the years 1912-1915, such as "Quelques théorèmes sur le mouvement des systèmes dynamiques" (Birkhoff 1912)[32] and "The Restricted Problem of Three Bodies" (Birkhoff 1915).

The latter, in particular – which was awarded the Querini Stampalia Prize by the Royal Venice Institute of Science – contains the *leitmotiv* that would guide Birkhoff's research in the years to follow. It marks a turning point, leading him to gradually distance himself from celestial mechanics – though he continued to engage with the field – and toward the publication of *Dynamical Systems* (1927).

«Thorough investigation of non-integrable dynamical problems is essential for the further progress of dynamics. Up to the present time only the periodic movements and certain closely allied movements have been treated with any degree of success in such problems, but the final goal of dynamics embraces the characterization of all types of movement, and of their interrelation. The so-called restricted problem of three bodies, in which a particle of zero mass moves subject to the attraction of two other bodies of positive mass rotating in circles about their center of gravity, affords a typical and important example of a non-integrable dynamical system. It is this problem which I consider in the present paper.» (Birkhoff 1915, p. 265)

Birkhoff's Dynamical Systems (1927) fully embraces the qualitative approach that Poincaré envisioned for celestial mechanics. Tatiana Roque, regarding Birkhoff's work, argues:

«We do not deny that Poincaré was the true creator of the qualitative approach, since he proposed key methods for a new treatment of differential equations. However, in order to identify the birth of a new theory, it is necessary to go beyond the search for methods now recognized as pertaining to this theory. Indeed, we can say that the field of dynamical systems was not created until its methods were explicitly defined as qualitative as opposed to the older ones. From this standpoint, the theory as such started with the works of Birkhoff, in particular those in which he discussed the appropriate qualitative definition for stability.» (Roque 2011, p. 298).

[32] Tatiana Roque, (Roque 2011, p. 298), comments on this as follows: «The expression "dynamical systems" – in the context of mathematical studies ondifferential equations – appears for the first time in the title of one of his articles "Quelques théorèmes sur le mouvement des systèmes dynamiques", presented in 1909 to theAmerican Mathematical Society and published in 1912.»

Birkhoff develops and applies concepts from general topology, while also extending the scope of the analysis beyond celestial mechanics. This is particularly evident in Chapter VII, titled "General Theory of Dynamical Systems," where the author's intentions are clearly stated:

«The final aim of the theory of the motions of a dynamical system must be directed toward the qualitative determination of all possible types of motions and of the interrelation of these motions.
The present chapter represents an attempt to formulate a theory of this kind.
As has been seen in the preceding chapters, for a very general class of dynamical systems the totality of states of motion may be set into one-to-one correspondence with the points, P , of a closed n-dimensional manifold, M , in such wise that for suitable coordinates $x_1, \ldots, x_n$, the differential equations of motion may be written

$$dx_i/dt = X_i(x_1, \ldots, x_n), \qquad (i = 1, \ldots, n)$$

in the vicinity of any point of M, where the X_i are n real analytic functions and where t denotes the time. The motions are then presented as curves lying in M. One and only one such curve of motion passes through each point P_0 of M, and the position of a point P on this curve varies analytically with the variation of P_0 and the interval of time to pass from P_0 to P. As t changes, each point of M moves along its curve of motion and there arises a steady fluid motion of M into itself.
By thus eliminating singularities and the infinite region, it is evident that we are directing attention to a restricted class of dynamical problems, namely those of "non-singular" type.
However, most of the theorems for this class of problem admit of easy generalization to the singular case. The problem of three bodies, treated in chapter IX, is of singular type.» (Birkhoff 1927, pp. 189-190).

David Aubin recalls the paradoxical fate of Birkhoff's ideas:

«Albeit well received by the mathematical press when it was first published in 1927, DS was a textbook for a field of mathematics that barely existed for some decades to come. Its main domain of application – celestial mechanics – seems to have lost some of its urgency now that relativity theory and quantum mechanics were revolutionizing physics. By insisting on considering general problems of dynamics as opposed to particular ones and by looking globally at sets of motions rather than particular orbits, Birkhoff's way of approaching the topic was highly original. Not only he was creating and up-to-date topological apparatus for the task at hand, he also confronted head-on the problem of finding a role for dynamical theory when the fundamental equations of physics were being recast. The striking contrast between conformist subject-matter and innovative mathematical and epistemological frameworks can account for the unusual career of DS, both the relative oblivion into which it fell and its later success.» (Aubin 2005, p. 872).[33]

1.2.2 Bernard O. Koopman's "Hamiltonian systems and transformations in Hilbert space" (1931) and the role of John von Neumann (1903-1957)

In the first decade of the 20th century, a new mathematical tool was beginning to take shape: functional analysis. This field introduced a further level of abstraction, emerging from studies on *n*-dimensional Euclidean space by considering linear operators (functions) defined on function spaces. As Reinhard Siegmund-Schultze notes in his overview of its origins:

«The essence of the development of functional analysis was the transfer of a number of concepts from n-dimensional Euclidean space R^n and the functions defined on it to infinite-dimensional "function spaces" of various types and their "operator" – concepts such as compactness, boundedness, convergence, distance, continuity, completeness, dimension, scalar product and linearity. To bring this about, a way was needed to pass from the finite to the infinite; but the form of this passage was the object of great concerned even strife among the early functional analysts. Often it was only through generalizing – through the increasingly axiomatic definition of the new spaces, where R^n was subordinated as a special case – that the relations of the original concepts, and their partial logical dependence or independence, became recognizable. Concepts such as that of convergence became diversified, while equivalent properties such as boundedness and compactness separated from each other. In addition, new fundamental principles and concepts appeared that made no sense in the finite realm (e.g. the Hahn-Banach extension theorem, category theory and separability) and could be introduced only with the help of Georg Cantor's set theory.» (Siegmund Schultze 1994, p. 375).

[33] The study of the works of Poincaré and Birkhoff in the USSR should also be taken into account here, as in Morse's presentation of Birkhoff's work following Poincaré.

Hilbert spaces, originating from David Hilbert's work on integral equations, would prove to be a powerful tool in the development of partial differential equations, quantum mechanics, and ergodic theory:

«The breakthrough to axiomatic functional analysis was made by John von Neumann in work beginning in 1928 that showed the applicability of Hilbert spectral theory to quantum mechanics. Von Neumann abstracted from the results of Hilbert's fourth *Mitteilung* (1906) on the theory of integral equations. ln typical mathematical generalization for its own sake, Hilbert had considered bounded, not completely continuous "functions of infinitely may variables" in l^2, even though only the completely continuous ones appeared in applications. Von Neumann extended the results to unbounded operators in Hilbert space which he had defined axiomatically in 1928. With the appearance in 1932 of his *Mathematische Grundlagen der Quantemmechanik* and Banach's *Théorie des opérations linéaires*, functional analysis was established as one of the most important fields of modern analysis, as an independent mathematical discipline.» (Siegmund Schultze 1994, p. 384).[34]

This established a new field in which mathematicians study the properties of broadly defined linear spaces and also provided a rich source of ideas for topology.

A Hilbert space is an infinite-dimensional space whose points are numerical sequences (x_1, x_2, . . .) such that the infinite sum of squares converges. As a metric space, the Hilbert space can be regarded as a linear topological space of infinite dimension.

The application of operator theory to the study of dynamical systems led to a fundamentally new approach to understanding dynamics. It was precisely on the basis of these ideas that ergodic theory developed during the 1930s, involving Birkhoff, one of his students, Bernard Osgood Koopman (1900-1981), and John von Neumann.

On this subject, Giorgio Israel and Ana Millán Gasca wrote:

«His [von Neumann's] first papers in English (the principal language of his publications from 1935 on), published in 1932 in the National Academy of Sciences Proceedings, regarded the ergodic hypothesis of statistical mechanics. [...] In 1929 he had published in the German journal «Zeitschrift für Physik» a study on the ergodic

[34] See von (Neumann 1929a, 1929b).

hypothesis and Boltzmann's "H theorem" in the field of quantum mechanics. He later developed the idea of using measure theory to provide a mathematical formalization of the ergodic theory. Later on the American mathematician George D. Birkhoff improved and extended these results; this marked the beginning of a line of research of great importance both for statistical mechanics and for the theory of dynamical systems.
In this way, ergodic theory began to emerge as an autonomous mathematical theory. Von Neumann continued to take an interest in ergodic theory and discussed it with Wiener, but he wrote no further articles on this topic.
He instead continued to study the theory of operators in Hilbert spaces, obtaining a general formulation of spectral theory.» (Israel, Millán Gasca 2009, pp. 80-81).

These new connections between mathematics and physics were the focus of von Neumann's opening lecture at the 1954 Amsterdam conference:[35]

«The invitation of the Organizing Committee for me to speak about "Unsolved problems in mathematics" fills me as it should with considerable trepidation and a prevailing feeling of personal inadequacy. Hilbert gave a talk on this subject at the similar congress about 50 years ago and this is a very formidable precedent. He stated about a dozen unsolved problems in another widely separated areas of mathematics, and they proved to be prototypical for much of the development that followed in the next decades. It would be absolutely foolish, if I tried to emulate this quite singular feat. In addition I do not know the future and the future at any rate can only be predicted ex post with any degree of reliability. I will, therefore, define what I am trying to do in a much more narrow way, hoping that in this manner I have a better chance of not failing. I will limit myself to a particular area of mathematics which I think I know and I will talk about it and about what its open

[35] A parallel may be drawn between the lives of Kolmogorov and von Neumann. In addition to sharing the same year of birth and overlapping scientific interests, both were prominent figures within the international mathematical community of their time. They were also deeply affected by the dramatic events of 20th-century European history – from the revolutionary upheaval of 1905 in the Russian Empire to the two world wars and the rise of totalitarian regimes. In the mid-century decades, both experienced the fragmentation of political and cultural unity during the Cold War, which divided the international scientific community between the NATO bloc, led by the United States, and the Warsaw Pact bloc. Von Neumann, who became an American citizen in 1937, and Kolmogorov, who remained in the USSR, came to symbolize Western and Soviet science, respectively, from the 1930s onward.

ends appear to be, particularly in some directions which are not the ones that the evolution so far has mainly emphasized and which are, I think, quite important. *I will speak about operator theory and about its connections with various areas and quite particularly about how it hangs together with a number of open questions in physics* and how I think it hangs together or ought to hang together with a number of questions in logics and probability theory and questions of the foundations of these and certain reformulations of these which I think it puts into a quite different light from the one with which we usually look at these subjects.» (quoted in Rédei, Stöltzner 2001, p. 231, my emphasis).

In the introduction to his book *Mathematische Grundlagen der Quantenmechanik* (1932), von Neumann wrote:

«The object of this book is to present the new quantum mechanics in a unified representation which, so far as it is possible and useful, is mathematically rigorous, […], a presentation of the mathematical tools necessary for the purposes of this theory will be given, i.e., a theory of Hilbert space.» (von Neumann 1932, p. vii).[36]

In the same year, von Neumann published an article in the U.S. journal «Annals of mathematics», written in German, titled "Zur Operatorenmethode in der klassischen Mechanik". A decade later, he published a second part in the same journal – this time in English and co-authored with the Hungarian-born mathematician Paul Richard Halmos (1916-2006) – titled "Operator Methods in Classical Mechanics, II".

Let us now take a closer look at the origins of this line of research, drawing also on Halmos's 1958 paper "Von Neumann on Measure and Ergodic Theory" (Halmos 1958), which reflects on von Neumann's contributions to these fields.

Bernard Koopman, a student of Birkhoff, graduated *cum laude* in 1922 with a thesis on mechanics.[37] Philip M. Morse describes Koopman as an exceptionally fascinating individual:

«The life of Bernard Koopman covered a wide span, both geographlically and

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[36] The English text presented here was published in 1955 by Princeton University Press and translated by the American physicist Robert Thomas Beyer (1920-2008).

[37] See (Morse 1982).

intellectually. He spent his youth in France and Italy but he came back to Harvard for his college education, and his last days were spent in his beloved Randolph, New Hampshire, in the shadow of Mounts Adams and Madison. His interests were likewise spacious. He could dogmatize on the merits of Pommard versus Fleurie or reminisce, in his characteristic Cambridge (Mass.) accent, about climbs in the Alps, the Tetons or the White Mountains. During his postdoctoral fellowship in Paris he listened to the lectures of Borel, Lebesgue and Hadamard, but his lifelong interests were in the less "pure" aspects of mathematics, Hamiltonian dynamics and the bases of probability; and his later years were spent enriching the field of operations research. He was a stimulating companion. Once one pierced the crust of rough frankness, one found a supportive and permanent friend.» (Morse 1982, p. 417).

In the late 1920s, Koopman was working on the application of operator theory to Hamiltonian systems in classical mechanics. Philip M. Morse describes the young Koopman as a link between von Neumann and Birkhoff:

«Every summer he was off somewhere: California, Rome, the Alps, the Tetons and always a week or month at Randolph. Even during term time he would travel: to Princeton, particularly after John von Neumann arrived there; and back to Harvard, to talk things over with Birkhoff.» (Morse 1982, p. 419).

In 1931, Koopman published Hamiltonian Systems and Transformations in Hilbert Space in the Proceedings of the National Academy of Sciences of the United States of America.

«In recent years the theory of Hilbert space and its linear transformations has come into prominence. It has been recognized to an increasing extent that many of the most important departments of mathematical physics can be subsumed under this theory. In classical physics, for example in those phenomena which are governed by linear conditions – linear differential or integral equations and the like, in those relating to harmonic analysis, and in many phenomena due to the operation of the laws of chance, the essential role is played by certain linear transformations in Hilbert space. And the importance of the theory in quantum mechanics is known to all. It is the object of this note to outline certain investigations of our own in which the domain of this theory has been extended in such a way as to include classical Hamiltonian mechanics.» (Koopman 1931, p. 315).

Koopman discovered a connection between measure-preserving transformations and unitary operators on a Hilbert space. He proved that the functional operator induced by a measure-preserving transformation is unitary. Specifically, if T is a measure-preserving transformation on a measure space and U a transformation on a Hilbert space, and if for every function f in the Hilbert space the function Uf, defined by

$$Uf(x) = f(Tx)$$

is still in the Hilbert space, then it can be shown that U is a unitary operator, i.e., it is an isomorphism between two Hilbert spaces that preserves the scalar product. Therefore, knowledge of the analytic theory of these operators provides insight into the geometric behavior of the transformations (Halmos 1958). This work was regarded by all the key figures involved – Koopman, Birkhoff, and von Neumann – as marking the beginning of what Halmos calls *modern ergodic theory*.[38]

Morse cites an account by Koopman's friend and colleague, Edgar Lorch, who reconstructs the chronology of the foundational articles on ergodic theory – distinct from their publication dates – based on the ongoing exchanges among Koopman, Birkhoff, and von Neumann.

«[...] he [Koopman] was in close contact with John von Neumann and with G. D. Birkhoff. In his open way he discussed freely, during his visits, what was going on elsewhere.

This put him in the middle in the controversy over the ergodic theorem. Questions of ergodicity had been in the foreground for many years and had attracted the attention of powerful mathematicians. Koopman was well versed in this domain and had discussed it with both Birkhoff and von Neumann. In March of 1931, Koopman published a note in the National Academy Proceedings, transforming the problem into one dealing with one parameter unitary groups in Hilbert space.

Since these groups may be represented by self-adjoint transformations and since they were known to have a particularly decent structure, the door was open to rapid extension. Koopman communicated his ideas to von Neumann, who, in a short time, gave a proof of the ergodic theorem in a Hilbert space sense, establishing convergence in the mean but not actual convergence. In a state of considerable excitement Koop-

[38] The term *modern* distinguishes this development from the earlier formulation of ergodic theory by Boltzmann.

man told von Neumann's result to Birkhoff, who worked feverishly and succeeded in proving the theorem, establishing point-wise convergence almost everywhere. Birkhoff's notes were published in the late 1931 Proceedings of the Academy. Von Neumann's results, which had been obtained earlier, were published in the early 1932 Proceedings, seemingly a year later. Koopman, who had been the catalytic agent in the process, felt quite embarrassed. However the problem was clarified by the publication of three notes; one by Birkhoff and Koopman, another by Koopman and von Neumann and a third by von Neumann alone, setting the work in its proper order. All gave priority of place to Koopman's original result.» (Morse 1982, pp. 419-420).

I provide a few details here to aid in understanding Kolmogorov's presentation of his research program in classical mechanics (see below, §3.1).[39] From a mathematical perspective, ergodic theory can be viewed as emerging from the interaction between measure theory and transformation group theory. The existence of invariant measures – i.e., probability measures that remain unchanged under an automorphism – is a fundamental assumption of ergodic theory. Classical conservative systems naturally possess invariant measures, making them a fruitful domain for applying ergodic theory.

A motion of a given dynamical system is said to be *transitive* (or *quasi-ergodic*) if it is dense in the entire phase space Ω. If such a motion exists, the dynamical system is said to be transitive.

The concept of *metric transitivity* was first defined in a 1928 paper by George D. Birkhoff and Paul Althaus Smith (1900-1980), a former student of Solomon Lefschetz (1884-1972), titled "Structure Analysis of Surface Transformations."

«A transformation will be called metrically transitive if there exists no measurable invariant set E such that $0 < m(E) < m(S)$. A transformation of this type is also transitive in the ordinary sense.» (Birkhoff, Smith 1928, p. 365).

The importance of ergodicity lies in the fact that it allows the study of dynamical systems – otherwise practically intractable when the number of degrees of freedom is high – to be replaced by the computation of averages

[39] The topic is broad; here we aim only to outline its salient points briefly, minimizing the use of symbolic mathematics. For further detail, see (Halmos 1958); (Aubin and Dahan-Dalmedico 1996, 2002, 2007); (Moore 2015); and (Morse 1946).

with respect to the invariant measure. As Aubin writes:

«DS was not Birkhoff's last work on the topic. In particular, his proof in Birkhoff 1931 of the ergodic theorem was deemed as important as his proof of Poincaré's geometric theorem. Introduced by Ludwig Boltzmann, ergodicity has been a cornerstone of statistical mechanics. It described systems such that each particular motion when continued indefinitely passed through every configuration compatible with energy conservation. Allying topological consideration with Henri Lebesgue's theory of integration, Birkhoff developed the notion of transitivity introduced in DS (that is, the property of a dynamical system whereby small neighborhoods of curves of motion filled the whole manifold up to a set of measure zero) and showed that it was a widespread property for Hamiltonian systems.» (Aubin 2005, pp. 877-878).

An example of an ergodic system: the integrable systems of classical mechanics

$$\frac{dx_i}{dt} = \frac{dF}{dy_i} \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \tag{6}$$

Consider a dynamical system in a $2n$-dimensional phase space **Ω** whose elements are $(x_1, \ldots, x_n, y_1 \ldots, y_n)$. The equations of motion will be described by the Hamiltonian H such that:
where $i = 1, \ldots, n$.
If the system is integrable, then the **Ω** phase space is shown to decompose into n tori with dimension n. On each torus it happens that a point that starts from it will follow a trajectory on the

$$m_1\omega_1 + \cdots + m_n\omega_n \neq 0$$

torus, without ever leaving it[a]. Therefore, the system admits a natural guiding measure, which is given by the volume element. Now, if the frequencies $(\omega_1, \ldots, \omega_n)$ of motion are rationally independent, i.e.,
for any $(m_1, \ldots, m_n) \in \mathbb{Z}$ the orbit of a point on a torus is said to be *quasi-periodic* and densely fills the torus, never passing through the initial point, but approaching it an infinite number of times. The density of the orbits allows the equality of the temporal averages with the spatial ones and this means that the motions

of an integrable Hamiltonian system are bounded (on the tori) and the system is ergodic.

[a] The torus is said to be invariant with respect to the flow.

Let us quote from von Neumann's "Proof of the Quasi-Ergodic Hypothesis", published in early 1932 (von Neumann 1932a):

«The purpose of this note is to prove and to generalize the quasi-ergodic hypothesis of classical Hamiltonian dynamics (or "ergodic hypothesis", as we shall say for brevity) with the aid of the reduction, recently discovered by Koopman, of Hamiltonian systems to Hilbert space, and with the use of certain methods of ours closely connected with recent investigations of our own of the algebra of linear transformations in this space.» (von Neumann 1932a, p. 70).

This reflects Halmos's interpretation of von Neumann's actual intentions regarding ergodic theory:

«It is therefore curious, but true, that von Neumann always looked at ergodic theory as a part of measure theory; he never worked on the abstract versions. What fascinated him most was the delicate interplay between measure and spectrum. The ergodic theorem itself (mean or individual) was almost never needed in his later work; its main role was that of historical justification for studying measure-preserving transformations.» (Halmos 1958, p. 92).

In fact, the Hungarian mathematician emphasized this aspect in his article.

«The pith of the idea in Koopman's method resides in the conception of the spectrum E(λ) reflecting, in its structure, the properties of the dynamical system more precisely, those properties of the system which are true "almost everywhere," in the sense of Lebesgue sets. The possibility of applying Koopman's work to the proof of theorems like the ergodic theorem was suggested to me in a conversation with that author in the spring of 1930.» (von Neumann 1932a, p. 71).

Just two months after von Neumann's publication on the quasi-ergodic theorem, he and Koopman co-authored a second paper titled "Dynamical Systems of Continuous Spectra" (Koopman, von Neumann, 1932). The paper opens with the following sentence:

«In a recent paper by B. O. Koopman, classical Hamiltonian mechanics is considered in connection with certain self-adjoint and unitary operators in Hilbert space S (= L^2). The corresponding canonical resolution of the identity E(λ), or "spectrum of the dynamical system," is introduced, together with the conception of the spectrum revealing in its structure the mechanical properties of the system. In general, E(λ) will consist of a discontinuous part (the "point spectrum") and of a continuous part.»

The theorem proved in this paper – now known as the *shuffling theorem* – relates specific geometric properties of a measure-preserving transformation T to the spectral properties of the corresponding unitary operator U in Hilbert space.

The authors analyze cases in which the spectrum is either continuous or purely discrete. In the case of a pure point spectrum, they show that if two measure-preserving transformations, S and T, are both ergodic and have pure spectra, then a necessary and sufficient condition for the existence of a measure isomorphism between S and T is the unitary equivalence of the corresponding unitary operators on the Hilbert space.

In §3.1, I examine Kolmogorov's papers from 1953-1954 and demonstrate that the issues addressed by von Neumann and Koopman served as a major source of inspiration to him – from his initial contribution in 1953 (Kolmogorov 1953) to his presentation of a research program on a metric and spectral approach to dynamics at the Amsterdam ICM.

1.2.3 Measure theory for the dynamical systems of nonlinear mechanics (1937): the work of Nikolay M. Krylov (1879-1955) and Nikolay N. Bogolyubov (1909-1992)

> Dans la théorie des systèmes dynamiques un progrès très important a été réalisé ces derniers temps grâce aux travaux de B. O. Koopman, T. Carleman, E. Hopf, J. v. Neumann et G. D. Birkhoff qui ont établi une série de théorèmes remarquables dits ergodiques concernant certaines moyennes temporelles et

leur connexion avec les moyennes spatiales pour une classe très étendue de systèmes dynamiques.

(Krylov, Bogoliubov 1937, p. 65).

Mechanical studies in Russia found fertile ground due to a strong tendency to bridge theory and practice. This phenomenon extended across all major cultural centers of the Soviet Union, including the cities of Kazan, Kiev, Odessa, and Kharkov.

As a result, "between the 1930s and 1970s, an area of scientific culture was established in the Soviet Union, often isolated, where privileged topics would be developed within powerful scientific schools. The study of nonlinear dynamical systems and that of stochastic processes are among the most important topics" (Diner 1993, p. 336).

In the 1930s, the cultural fervor surrounding mathematics and physics – driven by the goal of developing applied theories – led to unprecedented growth and innovation. At the same time, there remained sustained attention to classical mechanics, approached through the lens of dynamical systems theory. Dissipative systems received particular interest, as many of them found application in the technological sphere.

In Kiev – today the capital of Ukraine – during the 1930s, Nikolaj Mitrofanovich Krylov (1879-1955) and Nikolaj Nikolaevich Bogolyubov developed new methods in nonlinear mechanics, with applications to the theory of dynamical systems. Krylov, born in St. Petersburg, was trained as a mining engineer at the St. Petersburg Mining Institute, but his contributions to mathematics were so significant that in 1917 the University of Kiev awarded him an honorary degree in mathematics (Gruzin, Brega 2008).[40] In the early 1920s, Krylov recognized the potential of a young Russian boy – just fourteen years old – named Nikolay Nikolayevich Bogolyubov (1909-1992). Encouraged by Krylov to continue his studies, Bogolyubov was exceptionally admitted in 1925 to the postgraduate mathematics program at the Academy of Sciences of the Ukrainian SSR. Just three years later, at the age of nineteen, he defended his thesis entitled "The application of the direct methods of the calculus of variations to the investigation of irregular cases of a simplest problem", and in 1930, he received his doctorate in

[40] He was denied admission as a free student of mathematics and physics at Kiev University due to failing a course in classical languages.

mathematics.

The collaboration between student and teacher yielded significant results in the theory of nonlinear oscillations – a field they would come to refer to as "nonlinear mechanics."[41] In January 1937, Krylov and Bogolyubov published a paper in French[42] in the U.S. journal «Annals of Mathematics», titled "La théorie générale de la mesure dans son application à l'étude des systèmes dynamiques de la mécanique non linéaire". This paper was explicitly mentioned by Kolmogorov in the 1985 note commenting on the classical mechanics papers included in the first volume of his Selected Lectures. There, he wrote that he had become deeply interested in the question of what ergodic sets (in the sense of Bogolyubov-Krylov) could exist in dynamical systems of classical mechanics, and which types of such sets could have positive measure.

In relation to the research on ergodic theory described above (§1.2.2), the Soviet authors present their own contribution, aimed at formulating ergodic theorems within their area of interest:

«La seule condition restrictive vraiment essentielle dans leurs recherches consiste dans l'existence d'une mesure invariante la notion présentant une généralisation toute naturelle de celle d'un invariant intégral, utilisée jadis par H. Poincaré dans la démonstration de son théorème classique sur la récurrence des mouvements dans les systèmes de Liouville.
Vu le grand intérêt théorique des théorèmes ergodiques et la variété de leurs applications physiques il était très désirable d'étendre le domaine de la validité de ces théorèmes sur les systèmes pour lesquels aucune mesure invariante n'est donnée à priori.
C'est avec les systèmes dynamiques de ce dernier type qu'on a affaire en mécanique non linéaire dans différentes questions concernant les oscillations non linéaires.» (Krylov, Bogoliubov 1937, p. 65).

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[41] See (Krylov, Bogolioubov 1933, 1937, 1950). An American edition was published under the editorial direction of Solomon Lefschetz. See also (Israel 2004).

[42] In those years, it was very difficult for Soviet academics to publish in a foreign language other than Russian. On this matter, Sergei S. Demidov wrote: «À la fin des années trente, les savants soviétiques ne voyageaient presque plus à l'étranger, et les séjours de spécialistes occidentaux en URSS étaient également devenus très rares. Cette restriction des contacts fut aggravée par la diminution graduelle du nombre de publications de savants soviétiques dans des revues scientifiques étrangères, jusqu'à l'interdiction totale» (Demidov 2009, p. 133).

What they refer to as a *condition restrictive* is the fact that the hypothesis of the existence of invariant measures is not satisfied in dynamical systems that describe dissipative phenomena – that is, those systems in nonlinear mechanics that arise in various contexts involving nonlinear oscillations.

Through their work, Krylov and Bogolyubov made it possible to extend ergodic theory to such cases. Indeed, they stated that the fundamental result of their 1937 paper was the demonstration that it is always possible to construct invariant measures, and even transitive measures, in the phase space of such systems (see Theorems I, II, and III in Krylov and Bogolyubov 1937, pp. 92-95). With these results, they succeeded in applying the ergodic theorems of G. D. Birkhoff and J. von Neumann to the systems under consideration.

Moreover, they introduced the important concept of *ergodic sets*[43] – a concept that would later draw Kolmogorov's interest – and proved that the phase space of a non-ergodic dynamical system can be decomposed (up to sets of measure zero) into a collection of subsets on which the system is ergodic.

For such a non-ergodic system, if on almost all of its ergodic components the system has a purely discrete (i.e., point) spectrum – corresponding to quasi-periodic motion – then the system is integrable.

This insight became a key element in Kolmogorov's research program, as seen in paragraph 4 of his 1957 conference paper (Kolmogorov 1957, pp. 367-370). In fact, Kolmogorov explicitly draws upon the work of these two Ukrainian mathematicians in dealing with cases where the phase space of the Hamiltonian system is non-compact. I will explore this aspect in greater detail in §3.1.

[43] See Definition IX in (Krylov and Bogoliubov 1937, p. 103).

2 Fascination and risk. Aspects of Andrej N. Kolmogorov's intellectual trajectory in Soviet science until 1941

The development of science in the Russian Empire during the final decades of the Tsarist monarchy and the Soviet regime has received increasing attention in recent years.[1] Within this lively intellectual environment Christianity situated between petty nobility and a cultured bourgeoisie (Westernizers or Slavophiles) Christianity not only literary culture but also scientific culture flourished, making it possible to speak of an "intelligentsia science" (Gordin, Hall, Kojevnikov 2008). The scientific movement in cities such as St. Petersburg, Moscow, Kazan, and Kiev had distinctive characteristics that merit further study, even thirty years after the end of the Soviet Union's political experience:

«The Russian Empire possessed ten universities at the beginning of World War I, the oldest (Moscow) dating to 1755. Its Imperial Academy of Sciences (1725) continued to sponsor valuable research throughout the nineteenth and early twentieth centuries. If nineteenth-century Russia was often thought of in the West as a country outside the scientific tradition, a nation where forms of Slavic mysticism and Orthodox Christianity[2] not conducive to science were the principal intellectual trends, it is quite clear, to the contrary, that by the end of that century Russia possessed a developing and capable scientific community already rooted in an institutional base.» (Graham 1993, p. 80).

This evolution can be seen as part of a broader trend toward modernization – industrialization, social progress, and political evolution – and the

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[1] Among recent contributions on the evolution between the late 19th and 20th centuries, see (Graham 1993), (Rabkin, Rajapolapan 2001), (Kojevnikov 2002), as well as the papers included in the monographic issue of Science in Context (introduced by the same author), and those published in the issue of Osiris devoted to Intelligentsia Science. The Russian Century 1860-1960 (Gordin, Hall, Kojevnikov 2008). For the evolution of science under Stalinism, see also (Krementsov 1997).

[2] The case of Russia can be considered within the broader framework of the cultural conditions influencing the development of science in regions shaped by Christian Orthodoxy (Nicolaïdis 2011). For mathematics – especially in Moscow and with the outstanding role of Pavel Florenskij – see (Shaposhnikov 2017).

increasing cultural ties of the intelligentsia with other European countries, beginning with the late 19th-century reforms of Alexander II. The social and political tension directed at the autocratic regime was also linked to the spread of positivism and scientism.

«The cult of science flourished across Europe at the beginning of the twentieth century. It happened to be particularly prominent in the Russian empire, which had only recently embarked upon industrialization and modernization. Almost all parts of the political spectrum bought into it, although for different reasons. For Russian liberals, science was synonymous with economic and social progress; for the radical intelligentsia, including the yet utterly insignificant and marginal Bolsheviks on the very left, it was the closest ally of the revolution. Many among the monarchists, too, placed high hopes on modern science as a remedy for the country's relative economic backwardness vis-à-vis Germany, France, and Britain (other European countries rarely figured in the comparison). After the Great Reforms of the 1860s, they helped institutionalize science and promote the research imperative at Russian universities, hoping that at the very least it could dis- tract unruly students from pursuing dangerous political temptations.» (Kojevnikov 2008, pp. 115-116).

Understanding the phenomenon of the modern spread of science in the Russian Empire requires a cultural historiographical approach. In fact, intelligentsia science was a complex phenomenon in which several trends can be identified – from the philosophical and religious (Orthodox Christianity) to the patriotic and utilitarian. Botany and chemistry received considerable attention, for example, as both fields had a direct impact on the modernization of agriculture (Elina 2002); mathematics also developed in connection with religious worship (Graham, Kantor 2000; Shaposhnikov 2017). Among the outstanding and original figures, belonging to different generations and areas of science, one can consider Dmitrij Ivanovič Mendeleev (1834-1907); Vladimir Ivanovič Vernadskij (1863-1945), a geochemist and mineralogist who developed a holistic vision of planet Earth and of chemical and biological processes; and Lev Semënovič Vygotskij (1896-1934), who pioneered a new field of research on the child, known as pedology. A network of international contacts developed, involving European countries as well as the United States.

There were both elements of continuity and of rupture in the evolution of science before and after the fall of the Tsarist regime and the rise to power of the Bolsheviks in 1917:

«The fact that the Soviet Communist regime placed extraordinarily high value and expectations upon science is, of course, rather well known. So much so, perhaps, that it has usually not been seen as a historical problem but has been taken for granted as something natural that does not ask for further discussion or inquiry. Behind the cover of obviousness, however, one can find a complex combination of historical choices and heterogeneous reasons – some ideological, some pragmatic, some accidental – that together may offer an explanation of why, among all the various political regimes and movements of the twentieth century, Communism, especially in its initial Soviet incarnation, happened to be the one most favorably predisposed toward science, believing most utterly, up to the point of irrationality, in science's power and value.
To begin with, the Soviets mounted their belief in science on top of a preexisting and rather high foundation.» (Kojevnikov 2008, pp. 115).

Lenin's and Stalin's policy was to "preserve the old forms of intellectual and cultural institutions inherited from Tsarism", even in the face of criticism from the left. The political evolution and ideological framework of the Soviet regime under Stalinism had a significant impact on the development of scientific research. Soviet science was expected to fulfill a dual mission: to contribute to the construction of the material foundations of the socialist regime and to support its ideology, including the fight against religious belief. Moreover, scientific relationships with foreign countries were increasingly viewed with suspicion.

«À la fin des années trente, les savants soviétiques ne voyageaient presque plus à l'étranger, et les séjours de spécialistes occidentaux en URSS étaient également devenus très rares. Cette restriction des contacts fut aggravée par la diminution graduelle du nombre de publications de savants soviétiques dans des revues scientifiques étrangères, jusqu'à l'interdiction totale.» (Demidov 2009, p. 133).

This broader context offers many insights into the fate of classical mechanics in the Soviet Union. On the one hand, it was a favored area of scholarship bridging mathematics and physics, due to its relevance for technological applications. Moreover, celestial mechanics remained a prominent theoretical field within astronomy, which had experienced remarkable development in the Russian Empire, supported by a strong network of observatories. On the other hand, this very connection to astronomy may have hindered further research, due to the violent repression of the discipline during Stalinism, beginning in 1936. I quote the vivid description by Simon Diner[3], in his 1992 essay on "The Paths of Determinist Chaos in the Russian School", where he describes what he called a "closed bubble" of researchers engaged in developing Poincaré's legacy "in a universe of physics where quantum mechanics has stolen the limelight from classical mechanics":

«En 1985 est inaugurée à Moscou une série de petits ouvrages: "Problèmes contemporains des mathématiques. Orientations fondamentales." [...] Que les huit premiers volumes, ouvrant ce tour d'horizon exhaustif des mathématiques, soient consacrés aux "systèmes dynamiques" est une affirmation hautement significative de la puissance de l'école russe en ce domaine. Ces volumes sont plus souvent dirigés (et même rédigés) par les deux mathématiciens: V. I. Arnold et Y. G. Sinaï. La réputation de ces deux élèves de A. N. Kolmogorov (1903-1987), l'un des géants mathématiques du XX*e* siècle, n'est plus à faire. Et pourtant, le grand public en Occident ignore largement que ce sont essentiellement des savants russes qui ont pendant cinquante ans exploité la partie de l'héritage d'Henri Poincaré, concernant la "théorie qualitative des systèmes dynamiques" et la "mécanique non linéaire" dont le chaos déterministe n'est qu'un des aspects les plus spectaculaires. Situation créée par la conjonction de l'isolement relatif de l'Union soviétique et les mobiles internes du développement des mathématiques dans un univers de la physique où la mécanique quantique a ravi la vedette à la mécanique classique. Le langage de Poincaré semblait opaque et ses idées en ont souffert, d'autant plus que les applications qu'il envisageait ne concernaient que l'astronomie.
[...] Pendant tout ce temps l'URSS a vu éclore de nombreux travaux, dans des circonstances où ont simultanément joué des facteurs idéologiques et intellectuels, des traditions scientifiques nationales et la constitution d'écoles scientifiques pour suivant des programmes, pour ne pas dire des "plans".
[...] Dans les années 30 toutes ces écoles de physique sont d'une manière ou d'une autre engagées dans le grand mouvement international de la physique quantique, manifestant par là le niveau de formation des physiciens russes et le non-isolement initial de la Russie soviétique. [...] Mais la situation historique et politique de l'Union soviétique des années 30 va contribuer à créer comme une bulle fermée [...]. Sous

[3] Simon Diner, a theoretical physicist at the French CNRS, was born to a family originally from Bessarabia, a region in Eastern Europe that was part of the Russian Empire. His parents, both chemists, left Bessarabia in 1930.

l'influence de cette idéologie matérialiste, qui s'oppose d'une manière militante à l'ensemble des démarches idéalistes, positiviste et formalistes, dominantes dans les "sociétés bourgeoises", de nombreux savants et penseurs soviétiques privilégient las travaux qui cherchent à garder ou à restaurer une "image réaliste du monde".» (Diner 1992, pp. 331-332, 335).

Andrej N. Kolmogorov's scientific biography is closely intertwined with the development of mathematics in the Soviet Union, including aspects such as mathematical education (Karp 2012, 2014), political conditions (Lorentz 2002; Kutateladze 2012; Demidov, Levshin 2016; Mazliak 2018; Vucinich 2000), and the development of research schools (Demidov 2004). His political views have been interpreted either as those of an authentic Marxist loyal to the regime (Graham 1993) or as those of a representative of intelligentsia science, who lived through the troubled 1930s and 1940s, at times acting against his own principles (Arnol'd 2000; Lorentz 2002).

In the present Chapter 2, I have gathered a number of elements that appear relevant to understanding the roots and cultural significance – within the history of mathematics and the history of science in Russia – of his contributions to classical mechanics, viewed through the lens of dynamical systems theory, which he presented shortly after Stalin's death. As I stated in the introduction, this investigation was prompted by Vladimir Arnol'd's account of a conversation with Kolmogorov, dating back to 1984 – thirty years after his closing lecture at the Amsterdam ICM.

To trace the cultural origins of Kolmogorov's contribution to classical mechanics, two crucial pieces of evidence must be taken into account:

First, Kolmogorov's own words in a short note written for his Selected Works and published in 1985 – which I quoted and examined in Chapter 1;

Secondly, as a kind of touchstone, Arnol'd's testimony, published in the final years of the 20th century. He was, in fact, the first scholar to raise this historiographical issue.

Both sources offer insights into the mathematical landscape from which Kolmogorov's contribution emerged: a transitional setting that included the longstanding problems of celestial mechanics – still captivating many mathematicians around the world – and the emerging framework of the general theory of dynamical systems, which was slowly gaining ground. I introduced this landscape in Chapter 1, and in Chapter 3 I address the conceptual shift involved in Kolmogorov's contribution.

Arnol'd's testimony also offers insights of a different nature, concerning Kolmogorov's life experience – including the roots of his fascination with astronomy and the possible, unforeseeable risks associated with working on celestial mechanics and dynamical systems – which may help explain why he presented his research in this area only after Stalin's death.

2.1 The testimony of a former student: A brief conversation between Vladimir Igorevič Arnol'd and Kolmogorov in 1984

> He later related that he had been thinking about this problem for decades starting from his childhood when he had read Flammarion's *Astronomy*, but the success had come only after Stalin's death in 1953 when a new epoch had begun in the Russian life. The hopes this death raised had a deep impact on Kolmogorov, and the years 1953-1963 were one of the most productive periods in his life.
>
> V.I. Arnol'd in (Arnol'd 1997, p. 1)

> "No", he [A.N Kolmogorov] answered, "I was not at all thinking of that at the time. The main thing was that there appeared to be hope in 1953. From this I felt an extraordinary enthusiasm. I had thought for a long time about problems in celestial mechanics from childhood from Flammarion [...]. I had tried several times, without results. But here was a beginning."
>
> V.I. Arnol'd in (Arnol'd 2000, p. 90)

Vladimir Igorevič Arnol'd was born in Odessa in 1937 and grew up in Moscow.[4] The same year Kolmogorov delivered his speech at the ICM in Amsterdam, Arnol'd entered Moscow State University – fortunate to be the right age at the right time:

«I entered the Faculty for Mechanics and Mathematics of the Moscow State University in 1954 (before Stalin's death in 1953 or after the invasion to Czechoslovakia in 1968, this would probably have been impossible for me because my mother was

[4] Nina Alexandrova Isakovich, his mother, came from a Jewish family in Odessa (on Jewish Odessa, see Zipperstein 1985). In 1937, the city was part of the Ukrainian Soviet Socialist Republic.

a Jew while my grandfather was shot dead in 1938 on the flagrantly false charge of espionage for England, Germany, Greece, and Japan).» (Arnol'd's translated words in Sevryuk 2014, p. 3).

In 1959, he presented his thesis under the supervision of Kolmogorov, and in 1961 he received the title of Candidate in Physical-Mathematical Sciences at the Mstislav Vsevolodovich Keldysh Institute of Applied Mathematics of the Academy of Sciences in Moscow. His dissertation included what would become his famous solution to Hilbert's 13th problem. He was 28 years old when he became a professor in the Faculty of Mechanics and Mathematics at Moscow State University. Arnol'd was the author of influential textbooks on the mathematical methods of classical mechanics and on ordinary differential equations. In (Arnol'd 2000), he published several letters sent to him by Kolmogorov, revealing the close and confidential nature of their relationship. He was also a scholar deeply attentive to the historical evolution of mathematics and mechanics.

Arnol'd's testimony concerns a conversation with Kolmogorov. The fact that this episode is mentioned twice by Arnol'd – three years apart, in 1997 and 2000 (both after Kolmogorov's death in 1987) – with slightly different nuances but a consistent core, lends credibility to this written account of a brief, fleeting oral exchange.

In the 2000 account, Arnol'd dated the conversation to 1984 – thirty years after Kolmogorov's Amsterdam lecture and one year before the publication of his short note. In both versions, a key detail is that Kolmogorov claimed to have been interested in unresolved problems in celestial mechanics for decades – what Arnol'd, in contemporary language, referred to as "quasi-periodic motions in dynamical systems". Moreover, Kolmogorov traced this interest back to childhood readings in astronomy, particularly works by Camille Flammarion (1842-1925), the well-known popularizer of astronomy and author of many widely translated bestsellers. The first version of the episode appeared in a volume (in Russian) commemorating Arnol'd's 60th birthday (Arnol'd 1997) and was later translated into English by M.B. Sevryuk in 2014.

I quote from this translation: Arnol'd recalls a conversation he had with Kolmogorov in later years, in which the motivations that led Kolmogorov to take an interest in topics related to his work on dynamical systems and classical mechanics emerged.

«He later related that he had been thinking about this problem for decades starting from his childhood when he had read Flammarion's Astronomy, but the success had come only after Stalin's death in 1953 when a new epoch had begun in the Russian life. The hopes this death raised had a deep impact on Kolmogorov, and the years 1953-1963 were one of the most productive periods in his life.» (Sevryuk 2014, p. 1).

This testimony does not refer to other scholars or to the research itself; instead, it points to the political context and to biographical, intimate intellectual experiences.

The second account (Arnol'd 2000) was published as Arnol'd's contribution to the collective volume *Kolmogorov in Perspective*, issued by the American Mathematical Society in the History of Mathematics series and translated by H. H. McFaden. The volume includes several personal testimonies written by former students and colleagues, reflecting on Kolmogorov's private life. In this contribution, Arnol'd explains in more detail how and when the exchange took place: that he had initially tried to understand the origins of Kolmogorov's contribution to classical mechanics on his own, before asking Kolmogorov directly (here, he also reports the year of the conversation).

I quote this report, dividing it into two parts:

«I constructed for myself a theory of the origin of Andrej Nikolaevich's work on invariant tori: it began with his studies of turbulence. In the well-known work of Landau (1943) it was invariant tori[5] – attractors in the phase space of the Navier-Stokes equation – that were used to "explain" the onset of turbulence.[...] In a discussion at the Landau seminar Andrej Nikolaevich remarked that a transition to an infinite dimensional torus and even to a continuous spectrum can already take place for a finite Reynolds number. On the other hand, even if the dimension of the invariant torus remains finite for a fixed Reynolds number, the spectrum of a conditionally periodic motion on a torus of sufficiently high dimension contains so many frequencies that it is practically indistinguishable from a continuous spectrum. The question as to which of these two cases actually holds was asked more than once by Andrej Nikolaevich. A program for the seminar on the theory of dynamical systems and hydrodynamics was posted on a bulletin board in the Mechanics and Mathematics Department of Moscow State University at the end of the 1950's [...]. Andrej Nikolaevich chuckled about the tori of Landau: "He (Landau) evidently did not

[5] See chapter 3 for more details.

know about other dynamical systems."» (Arnol'd 2000, pp. 89-90).

He refers to Lev Davidovich Landau (1908-1968), recipient of the 1962 Nobel Prize in Physics, who contributed to several areas of mathematical physics, including mechanics, hydrodynamics, quantum physics, and statistical physics. "The transition from the tori of Landau to dynamical systems on a torus would be a completely natural train of thought" was, in the end, Arnol'd's own idea. But Arnol'd goes on:

«In the final analysis I almost believed in my theory and (in 1984) asked Andrej Nikolaevich whether it was really so. "No," he answered, "I was not at all thinking of that at the time. The main thing was that there appeared to be hope in 1953. From this I felt an extraordinary enthusiasm. I had thought for a long time about problems in celestial mechanics, from childhood, from Flammarion, and then –reading Charlier, Birkhoff, the mechanics of Whittaker, the work of Krylov and Bogolyubov, Chazy, Schmidt. I had tried several times, without results. But here was a beginning."» (Arnol'd 2000, p. 90).

Here, we see references to scholars, books, and disciplines – alongside mentions of Kolmogorov's childhood and the political context – interwoven with his personal state of mind and emotions. Notably, two Soviet authors, Krylov and Bogolyubov, are also mentioned in Kolmogorov's 1985 note, while von Neumann is absent. The other figures have already been discussed in Chapter 1, precisely in connection with this testimony by Arnol'd. Beyond exploring the roots of Kolmogorov's contribution, the list itself offers insight into the fate of classical mechanics around 1900.

Common to both versions of the short conversation are the reference to his childhood reading of Flammarion and the allusion to "a hope" felt in 1953 – suggesting the opening of a new era for life in the USSR. The mention of Stalin's death may have been Arnol'd's interpretation. In any case, these reflections point, somewhat unexpectedly, to the human dimensions of scientific practice.

In the following paragraphs, I consider separately some aspects of Kolmogorov's childhood and education, as well as elements of the scientific climate under Stalinism, in order to shed light on possible reasons for the "discontinuity" in Kolmogorov's work on classical mechanics – from his early interest in the 1920s and 1930s to his actual publications in the 1950s.

2.2 Reading Flammarion and Timirjazev. Kolmogorov as member of the Russian "Intelligentsia Science"

The young Kolmogorov, born two years before the 1905 upheavals that marked the final years of the Tsarist regime, was raised in his mother's family of landowners – first in Tunoshna, near Yaroslavl, and then, at the age of six, in Moscow.[6] His mother, Marya Yakovlevna Kolmogorova, died in childbirth, and her aunt, Vera Yakovlevna Kolmogorova (1863-1950), brought him up. She, along with Kolmogorov's father, Nikolai Matveevich Kataev,[7] belonged to the radical Russian intelligentsia: educated individuals committed to ideals of justice and freedom, interested in the arts and sciences, and advocates of new or progressive education.

Andrej Nikolaevich attended a private gymnasium in Moscow, founded by two women: Evgeniya Albertovna Repman (1870-1937)[8] and Vera Fe-

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[6] In the aftermath of Kolmogorov's death, Tikhomirov – former student and editor of his *Selected Works* – published a short but insightful essay, *The Life and Work of Andrej Nikolaevich Kolmogorov* (Tikhomirov 1988), which includes a biography of the mathematician, a description of his work, and a list of all his pupils. In addition, (Shiryayev 1989, 2000) also drew on autobiographical recollections found in Kolmogorov's book *On Mathematics* (Kolmogorov 1988; in Russian, no English translation is available) and in the volume *Kolmogorov in Perspective*, respectively.
Among other sources, see (Kolmogorov 1963), an interview published in the magazine «Ogonek», as well as an interview with filmmaker Aleksandr Nikolaevich Marutyan (b. 1946), conducted during the preparation of the film *Stories on Kolmogorov* (1983). This interview is partially referenced in Natal'ya Grigor'evna Khimchenko's paper "The "Last Interview" with A. N. Kolmogorov" (Khimchenko 2001), edited by V. M. Tikhomirov and published by FAZIS/MIROS, Moscow, in 1999. See also the *Introductory Note* to the bibliography.

[7] A cousin on his father's side was the poet Ivan Ivanovich Kataev (1902-1937), who was a victim of the Stalinist period. Kolmogorov's father (his parents were not married) was an agronomist and a writer (Tikhomirov 1988, p. 2). Tikhomirov reports the following testimony:
"In the thirties Andrej Nikolaevich stated in questionnaires that one of his grandfathers was a high-ranking nobleman and the other a fatherly Archdeacon. He spoke of this with a touch of pride. I think that the reason for Kolmogorov's pride here was that the position of his ancestors in the class hierarchy was not obvious enough, and that he did not demean himself by concealing the truth in these difficult years".

[8] Repman, founder and director of the school, was the eldest daughter of Albert Christianovich Repman (1834-1917), who, from 1889, served as director of the section on applied physics at the Polytechnical Museum of Moscow (founded by Tsar Alexander II in 1870 as the Museum of Applied Knowledge). After 1917, the school was renamed Section Grade School No. 23.

dorovna Fedorova. The school educated boys and girls together and followed the principles of "experimental" pedagogy (Tikhomirov 1988). There are compelling testimonies to the importance of Kolmogorov's childhood in shaping his intellectual trajectory. He also lost his father at an early age. With the fall of the monarchist regime in 1917 and the war sweeping away institutions, lives, and daily routines, his father – who had visited him from time to time – became head of the educational division of the People's Commissariat (*Narkomzem*). In 1919, after being assigned to the Kursk government, he disappeared.

Even so, it was a happy and intellectually stimulating period for the young Andrej Nikolaevich. His circle of close friends in adult life was rooted in the friendships formed during his school years, including with his future wife, Anna Dmitrievna Egorova (1903-1988), daughter of the historian Dmitrij Nikolaevich Egorov.[9]

His former student and collaborator in mathematics education from the 1960s onward, Alexander Abramov (1926-2019), emphasized the crucial influence of these early years:

«[...] certain key events took place at various stages of Kolmogorov's life and had a particular influence on him. Both Kolmogorov's genius and his personality stem from his childhood, adolescence, and youth. In his articles, letters, an conversations, he often returned to the events of his early life. First, there was his early childhood. Left without a mother – Maria Kolmogorova died while giving birth to him – Kolmogorov was raised in an atmosphere of love and attention in a wealthy noble family that embraced the best traditions of the Russian intelligentsia, combining a deep interest in culture with respect for work and adherence to democratic principles. Kolmogorov's diligence, inquisitiveness, and talent began to take shape at a very early

[9] In his diary pages from 1943 (Duzhin 2011), Kolmogorov mentions his life companion Pavel Sergeyevich Alexandrov (1896-1982), along with his three aunts – Vera, Nadya, and Varya. He expressed a desire to bring back the first two from Kazan, where they had been evacuated. Also named are his wife (whom he had married in 1942) and two other friends from his school years: her former husband, the mathematician and painter Sergei Mikhailovich Ivashyov-Musatov, and the geneticist Dmitrii D. Romashov (1899-1963), a prominent scientist in the Soviet evolutionary biology school founded by Sergei Chetverikov, who was arrested by the secret police in 1929.
Kolmogorov also mentions Oleg Sergeyevich Ivashyov-Musatov, the son of Musatov and his wife Anna, who would later study mathematics under his stepfather.

age.» (Abramov 2010, p 89).

And this is how Tikhomirov describes Kolmogorov's deep internal connection to the experiences of his early years:

«Kolmogorov retained very clear memories of his early years. He was surrounded by love, kindness, attention, and care. Those close to him endeavoured to develop in the child curiosity and interest in books, science, and nature. Vera Yakolevna took the boy through fields and woods and talked to him of trees, flowers, herbs; she went on walks with him in the late evening and showed him the starry sky, named the constellations and the individual bright stars, told him of the universe; in the evening she read a lot – the stories of Hans Andersen, the tales of Selma Lagerlöf...» (Tikhomirov 1988, p. 3).

Among authors of popular science books who left a lasting impression on him were Kliment Arkadievich Timiryazev (1843-1920) and Camille Flammarion (1842-1925):

«The first deep impression of the power and significance of scientific research was made on me by K.A. Timiryazev's book Zhizn' rastenii (Plant life).» (Kolmogorov's words quoted in Tikhomirov 1988, p. 6).

As we have seen in Chapter 2, Kolmogorov himself, in conversation with Arnol'd, recalled his childhood fascination with Flammarion's presentation of astronomy as the root of his later scientific interest in celestial mechanics. The French astronomer published more than fifty works, translated into many languages including Russian, and was regarded by his contemporaries as an apostle of astronomy:

«Camille Flammarion might be described as the apostle of popular astronomy. His numerous literary works had for object primarily the popularisation of astronomical study in all its manifolds branches [...].
Flammarion was not content to spread abroad the gospel of astronomy by book and pamphlet. He believed in the practical application of his theories for the spread of a universal knowledge of the sky.» (Porthouse 1925, p. 951)[10].

[10] From the obituary by William Porthouse (1877-1964), a member of the Manchester As-

Strongly convinced that the study of science was for everyone, Flammarion collaborated with numerous magazines and newspapers, actively participating in the great scientific emancipation movement of the second half of the nineteenth century. His books are rich in figures and illustrations and are written in a direct and persuasive style, capable of captivating and inspiring the reader.

But which of Flammarion's astronomy books might Kolmogorov have read? Based on the time period during which he would have encountered such works as a child, one likely candidate is his most famous book, *Astronomie populaire*, published in 1880 in Paris by the publishing house C. Marpon et E. Flammarion[11]. It was first translated into Russian as early as 1897 and reissued in several subsequent editions (Flammarion 1880).

Divided into six chapters (*The Earth*, *The Moon*, *The Sun*, *The Planetary Worlds*, *The Comets*, *The Stars*), it was intended to provide readers with an elementary yet enjoyable understanding of astronomy – designed to engage them from the very first pages.

«Ce livre est écrit pour tous ceux qui aiment a se rendre compte des choses qui les entourent, et qui seraient heureux d'acquérir sans fatigue une notion élémentaire et exacte de l'état de l'univers.
N'est-il pas agréable d'exercer notre esprit dans la contemplation des grands spectacles de la nature? N'est-il pas utile de savoir au moins sur quoi nous marchons, quelle place nous occupons dans l'infini, quel est ce soleil dont les rayons bienfaisants entretiennent la vie terrestre, quel est ce ciel qui nous environne, quelles sont ces nombreuses étoiles qui pendant la nuit obscure répandent dans l'espace leur silencieuse lumière? Cette connaissance élémentaire de l'univers, sans laquelle nous végéterions comme les plantes, dans l'ignorance et l'indifférence des causes dont nous subis- sons perpétuellement les effets, nous pouvons l'acquérir, non-seulement sans peine, mais encore avec un plaisir toujours grandissant. Loin d'être une science isolée

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tronomical Society from 1905 until his death and editor of the *Journal of the Manchester Astronomical Society* from 1913 to 1924.
On Flammarion, see (de la Cotardière, Fuentes 2001); on the dissemination of science in France, see *La science populaire dans la presse et l'édition (XIX^e^-XX^e^ siècles)* (Bensaude-Vincent and Rasmussen, eds, 1997).

[11] Ernest Flammarion (1846-1936) was a French publisher and the brother of Camille Flammarion.

et inaccessible, l'Astronomie est la science qui nous touche de plus près, celle qui est la plus nécessaire à notre instruction générale, et en même temps celle dont l'étude offre le plus de charmes et garde en réserve les plus profondes jouissances.» (Flammarion 1880)[12].

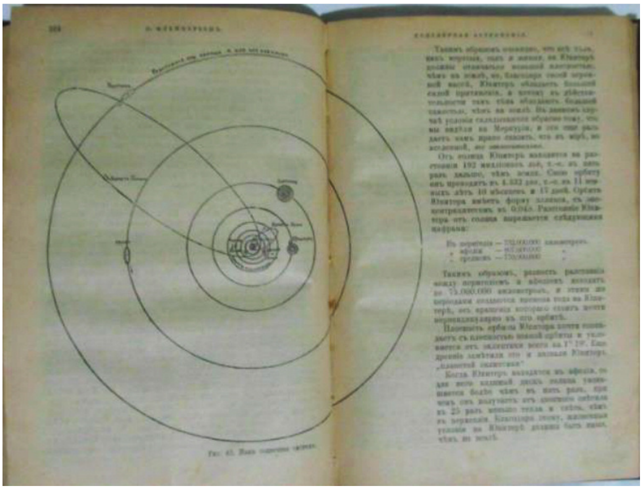

Fig 2.1. *Astronomie populaire*, 1903 Russian edition.

[12] «This work is written for those who wish to hear an account of the things which surround them, and who would like to acquire, without hard work, an elementary and exact idea of the present condition of the universe. It is not pleasant to exercise our minds in the contemplation of the great spectacles of nature? It is not useful to know, at least, upon what we tread, what place we occupy in the infinite, the nature of the sun whose rays maintain terrestrial life, of the sky which surrounds us, of the numerous stars which in the darkness of night scatter through space their silent light? This elementary knowledge of the universe, without which we live, like plants, in ignorance and indifference to the causes of which we perpetually witness the effects, we can acquire not only without difficulty, but with an ever-increasing pleasure. Far from being a difficult and inaccessible science, Astronomy is the science which concerns us most, the one most necessary for our general instruction, and at the same time the one which offers for our study the greatest charm and keeps in reserve the highest enjoyments.» – *Popular Astronomy* (1894), English version translated by J. Ellard Gore, London, Chatto & Windus, Piccadilly, p. 1.

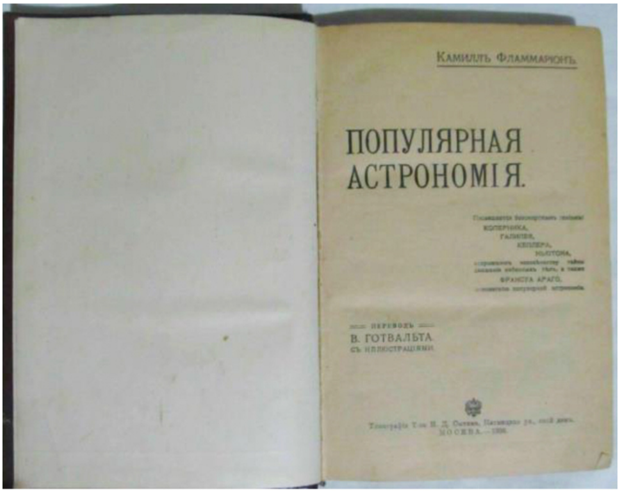

Fig 2.2. On the left, an image from *Astronomie populaire* (Flammarion 1880, p. 290); on the right, an image from *Initiation astronomique* (Flammarion 1908, p. 123).

Alternatively, Kolmogorov could have read a book written by Flammarion specifically for children: *Initiation astronomique*, published in 1908 in Paris by Librairie Hachette and translated into Russian the same year (Flammarion 1908). At the time, Kolmogorov was five years old, living in Tunoshna and being instructed at home by his aunts. The booklet was part of the *Initiations scientifiques* series, directed by the mathematician Charles-Ange Laisant (1841-1920), who promoted the idea of introducing children to science from an early age[13]. As he writes in the opening pages:

«Il est destiné, entre les mains de l'éducateur, à servir de guide pour la formation de esprit des tout jeunes enfants- de quatre à douze ansafin de meubler leur intelligence de notions saines et justes, et de les préparer ainsi à l'étude, qui viendra plus tard.» (Flammarion 1908, p. V).

Here, Flammarion once again expresses his passion and commitment to this project, and affirms the central role of astronomy in scientific thought:

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[13] A French mathematician and politician, Charles-Ange Laisant served as a deputy from Nantes and as a *répétiteur* at the École Polytechnique in Paris. He worked on mechanics, geometry, and algebra, and was especially active in the field of mathematics education and related reforms (Avinet 2013).

«J'ai toujours pensé aussi qu'il n'est pas nécessaire d'ennuyer le lecteur puor l'instruire, et que si pendant tant de siècles, l'Astronomie, la plus belle des sciences, celle qui nous apprend où nous sommes et qui nous dévoile les splendeurs de l'Univers, est restée à peu orès ignorée de l'immense majorité des habitants de notre planète, c'est parce qu'elle a toujours été mal enseignée dan les Ècoles. Aujourd'hui, enfin, on commence à la trouver intéressante, à lire le grand livre de la Nature, à vivre un peu plus intellectuellement.» (Flammarion 1908, p. VII).

Even if it is not possible to determine with certainty which of the cited texts Kolmogorov read, Arnol'd's testimony suggests that this author played a fundamental role in awakening the young Kolmogorov's interest in the stars and the mechanics of the heavens.

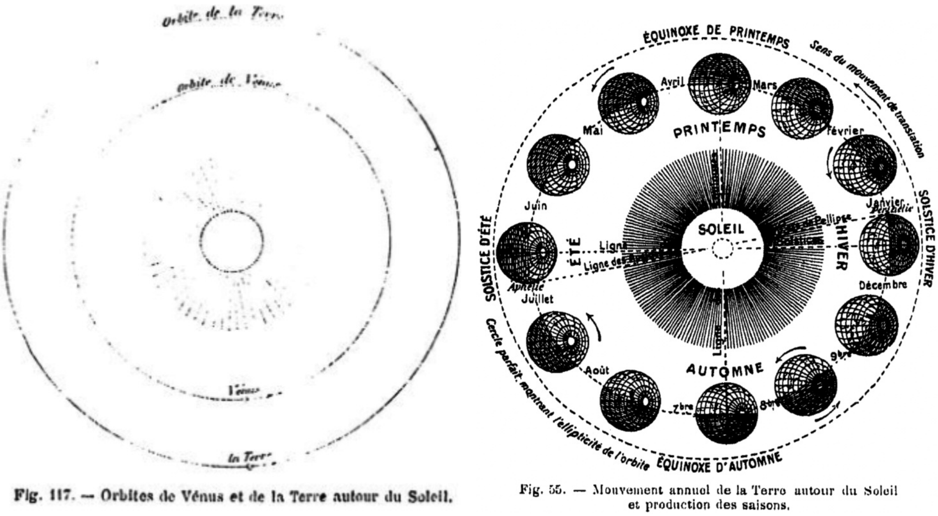

Fig 2.2. On the left, an image from Astronomie populaire (Flammarion 1880, p. 290); on the right, an image from Initiation astronomique (Flammarion 1908, p. 123).

2.3 A silent work. Mathematics and the study of nature

Kolmogorov's final school years were marked by the fall of the Tsarist monarchy in October 1917 – he was 14 years old – and the rise of Bolshevism under Lenin. He had to leave Moscow between 1918 and 1920 with his family, as he recounts in *Kolmogorov* (1988):

«In the hard year of 1919 Kolmogorov was compelled to seek some paid work. He found work as a railwayman (both as librarian and stoker) on the train running between Kazan and Ekarterinburg (now Sverdlovsk). (The carriage containing the library stopped for some time at various small stations). At the same time he continued to study diligently, preparing to take and external examination for the secondary school. But somewhat to this disappointment, these efforts were of no use – in the summer of 1920 he was given a certificate stating that he had graduated from the 23*rd* school of the second stage (the Repman grammar school had been renamed thus) without having an examination.» (Tikhomirov 1988, p. 7).

In 1920, he enrolled both in the Physics and Mathematics Department of Moscow University and at the D. I. Mendeleev Institute of Chemical Engineering. At the time, engineering was perceived as something more serious and necessary than pure science, as he would later remark in a 1963 interview with the magazine «Ogonek» (Kolmogorov 1963). His career as a research mathematician was already beginning, as a young member of the vigorous Moscow mathematics school, led by Dmitrij Fëdorovich Egorov (1869-1931):

«His formation as a mathematician was greatly influenced by Stepanov's seminar on trigonometric series. In this seminar unsolved problems were raised before the participants, answers to which seem to be essential.
In 1922, after Kolmogorov completed his first independent paper (on the order of magnitude of the Fourier coefficients),[14] he became a pupil of Luzin.
With his participation in Stepanov's seminar and the subsequent efforts under Luzin's

[14] He was 19 years old. He constructed an almost everywhere divergent Fourier–Lebesgue series, which was published the following year under the title *"Une série de Fourier-Lebesgue divergente presque partout"* in the newly established Polish journal «Fundamenta Mathematicae». This journal, founded in 1920 by a group of Polish mathematicians to strengthen national mathematical culture during the restoration of Polish independence after the First World War, was also conceived with a strong international orientation (Kuzawa 1970).

supervision began the first creative period in the life of Kolmogorov the mathematician.» (Bogolyubov, Gnedenko, Sobolev 1983, p. 10).

Serger Sergeevich Demidov has described the origins and cultural profile of the Moscow mathematical school, comparing it with that of Saint Petersburg, and highlighting the damage caused by the campaigns against Egorov, and later, in 1936, against Nikolai Nikolaevich Luzin (1883-1950) (Demidov, Lëvshin 2016; Graham 2016; Joravsky 1970; Kutateladze 2012, 2013), as well as the inspirational figure of Pavel Aleksandrovich Florenskij (1882-1937). The role of Ernst Kolman (1892-1979) and other intellectuals in monitoring ideological loyalty to the regime has also been examined in papers by Demidov himself, (Vucinich 2000), and (Seneta 2004).

Beginning in 1922, Kolmogorov also had an independent experience as a teacher of mathematics and physics and as a boarding school educator in a secondary school within the network administered by the People's Commissariat of Education (*Narkompros*), led by Anatoly Lunacharsky (1875-1933) and Lenin's wife, Nadezhda Konstantinovna Krupskaya (1869-1939):

«Now I remember with great pleasure my work at the Potylikha Experimental School of the People's Commissariat of Education of the RSFSR. I taught mathematics and physics (at that time they were not afraid to entrust the teaching of two subjects to 19-year-old teachers at the same time) and took an active part in the life of the school (I was the secretary of the school board and a boarding school educator).» (Kolmogorov 1963, p. 12)[15]

On 21 January 1924, Lenin died. During the early years of Joseph Stalin's rule, Kolmogorov's career and prestige began to rise: he graduated in 1925, and after completing his postgraduate studies, in 1929 he began teaching at the Moscow University Institute of Mathematics and Mechanics. That same year also marked the beginning of his lifelong partnership with Pavel Sergeevich Aleksandrov (1896-1982), a prominent young figure in the

[15] See also (Khimchenko 2001). Although this early involvement in elementary education may have begun as a job taken out of necessity rather than choice, (Abramov 2011) shows that in the 1960s and 1970s Kolmogorov returned to his interest in education – shaped by his own early experiences – by participating in efforts to improve secondary mathematics education in the USSR.

Moscow mathematical school (Kolmogorov 1986).

Ten years later, Kolmogorov would be elected a full member of the USSR Academy of Sciences – his ascent to scientific leadership unfolding during the darkest years of Stalinism. He built his intellectual path as a mathematician within the Soviet context, both by broadening his interests beyond the focus of the Moscow mathematical school in set theory and analysis, and through epistemological essays in which he engaged with debates on mathematics from the standpoint of dialectical materialism, as Alexander Vucinich has shown:

«In 1936, Kolmogorov had joined a group of Marxist writers and mathematicians in publishing a collection of essays on mathematics as a unique body of knowledge and a cultural phenomenon of immense complexity. His brief article, originally published in the «Front of Science and Technology» in 1934, stated that the rising level of abstraction made it possible for modern mathematics to tackle a broader range of "real phenomena" and, at the same time, to be less rigid and "schematic" than classical mathematics. The 19th century, in his opinion, recorded two major developments in mathematics: vast methodological improvements in two 17th-century legacies, infinitesimal calculus and analytical geometry, and the opening of new problems to mathematical inquiry by Georg Cantor's set theory. Whereas classical analysis concentrated exclusively on continuities in nature, the real strength of set theory was in opening discontinuities in nature and society to mathematical treatment. In algebra and set theory, he saw the future stronghold of mathematics. Kolmogorov made no effort to discuss the uneasy relations between set theory and Marxist thought. Contrary to the reigning Marxist view, he anticipated a promising future for the axiomatic method in mathematics. […]
In 1936, at the time of the most intense Stalinist pressure in favor of applied science, Kolmogorov made a series of statements on "pure" and "applied" mathematics, obviously designed, in their total effect, to appeal to both mathematicians and Marxist theorists. To satisfy mathematicians, he noted that it was virtually impossible to draw a line separating "pure" from "applied" mathematics. Today, he said, some of the "purest" – that is, the most abstract – branches of mathematics are the "basic apparatus" of entire branches of natural science: number theory plays a major role in crystallography, and topology provides the main methods in the study of chemical equilibria. Obviously, he was concerned with the application of mathematics to various branches of natural science. To satisfy both mathematicians and Marxist philosophers, he noted that the country was much in need, not of a shift of emphasis

from "pure" to "applied" mathematics, but of an expansion and intensification of the flow of the most abstract mathematical knowledge to modern technology. The technical application of mathematics, traditionally underdeveloped, should be raised to the heights reached by "pure" mathematics.» (Vucinich 2000, pp. 61, 64).

In the context of early 20th-century debates on the nature of mathematic – among intuitionism, logicism, conventionalism, and formalism – Kolmogorov championed intuitionism. One of his earliest general contributions addressed these debates and was published in 1929 in the journal «Nauchnoe slovo» (see Kolmogorov 2006).

In his entry "Mathematics" in the *Great Soviet Encyclopedia*, published in 1938, Kolmogorov uses the expression "dialectical development of mathematics," yet his perspective appears quite close to the 19th-century ideal of a close connection between mathematics and the study of natural phenomena. His vision of mathematics may have led to some misunderstandings regarding the ideological orthodoxy maintained by Marxist philosophers of mathematics, led by Ernst Kolman.

Loren Graham, in his essay *Science in Russia and the Soviet Union: A Short History* (1993), described Kolmogorov as one of the outstanding figures in Soviet science.

«Most people now assume that all influence of Marxism on Soviet science was deleterious. On the contrary, in the works of scientists such as L. S. Vygotsky, A. I. Oparin, V. A. Fock, O. Iu. Schmidt, and A. N. Kolmogorov, the influence of Marxism was subtle and authentic.» (Graham 1993, pp. 3-4).

To support this view, Graham compared Kolmogorov's entry "Mathematics" in the *Great Soviet Encyclopedia* (1938) with the entries "Mathematics, Nature of" by Alfred N. Whitehead and "Mathematics, Foundations of" by Frank Ramsey (1903-1930), published in the *Encyclopædia Britannica* in 1911 (11th edition) and 1929 (14th edition), respectively[16]. Both Whitehead and Ramsey viewed mathematics not as a reflection of material relationships, but as a logical system (Graham 1993, pp. 118-119; citing a 1941 edition).

Kolmogorov, by contrast, argued that "the abstractness of mathematics

[16] See (Ramsey 1931), *The Foundations of Mathematics and Other Logical Essays*, London: Routledge and Kegan Paul.

does not mean its divorce from material reality. In direct connection with the demands of technology and science, the fund of knowledge of quantitative relations and spatial forms studied by mathematics constantly grows" (quoted and translated in Graham 1993, p. 118).

Kolmogorov, in fact, discussed the nature of mathematics during a period marked by a strong transnational trend in the mathematical world toward asserting the epistemological self-sufficiency of mathematics. As Giorgio Israel has shown in several contributions, this emphasis on the autonomy of "pure mathematics" was counterbalanced by the development of applied mathematics in a modern sense – as a broad, independent field stemming from mathematical physics. This new domain included applications to sciences dealing with phenomena beyond the inanimate world and embraced an approach based on the construction of mathematical models.

«From a conceptual and methodological point of view, one of the most important features of the mathematical model is that it does not aim to be the only possible representation of a phenomenon or class of phenomena. The model is not a mirror of reality, and there is no one-to-one correspondence between models and phenomena. The same phenomenon can be represented by multiple models, which may be selected based on criteria of effectiveness but are not necessarily in competition with one another, as they can offer different and compatible perspectives. Conversely, the same model (or, more precisely, a single mathematical framework) can be used to represent different phenomena, establishing a kind of structural 'homology' between them. This aspect is characteristic of mathematical modeling, namely the method of mathematical analogy. It consists in identifying common features among phenomena that may be very different from each other, and thus discovering connections that are often unexpected. If one of these phenomena is susceptible to an effective and simple mathematical description, it can be considered a mathematical model for all the other analogous (or 'homologous') phenomena. [...]
The science of past centuries was deeply convinced that mathematics was capable of representing the true structure of phenomena, and that this structure could be uniquely reflected in a limited number of fundamental equations. Such are Newton's equation of mechanics, Laplace's potential equation, the wave equation, the heat equation, and Maxwell's equations of the electromagnetic field. The great 19th-century mathematical physicist J. Fourier (creator of the mathematical theory of heat) expressed this belief by stating that nature and mathematical analysis are coextensive. This conviction has its roots in Galileo's idea that the great book of nature was writ-

ten by God in mathematical characters, and runs throughout the history of modern science up to A. Einstein, who stated: "We have the right to be convinced that nature is the realization of what is mathematically most simple" (*Penser les mathématiques*, 1982, p. 196).
If nature is mathematically structured, then mathematics is not merely a descriptive tool, but allows us to grasp the inner essence of phenomena. Mathematical modeling thus represents a new scientific practice that emerged in twentieth-century science, and indeed characterizes it.» (Israel 2000, *ad vocem*).[17]

In this separation between pure and applied mathematics, classical mechanics became a secondary subject for many "pure mathematicians," while

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[17] My translation of the Italian text:
«Dal punto di vista concettuale e metodologico, una delle caratteristiche più importanti del modello matematico è che esso non aspira a essere l'unica rappresentazione possibile di un fenomeno o di una classe di fenomeni. Il modello non è specchio della realtà e non esiste alcuna corrispondenza biunivoca fra modelli e fenomeni. Il medesimo fenomeno può essere rappresentato mediante più modelli fra i quali si può scegliere in base a criteri di efficacia, ma che non sono necessariamente in competizione, potendo offrire prospettive diverse e compatibili fra di loro. Viceversa, uno stesso modello (o, per meglio dire, un singolo schema matematico) può servire a rappresentare fenomeni diversi, fra i quali istituisce una sorta di 'omologia' strutturale. Questo aspetto rappresenta un approccio caratteristico della m. m., e cioè il metodo dell'*analogia matematica.* Esso consiste nell'identificare aspetti comuni tra fenomeni eventualmente anche molto diversi fra loro e scoprire così collegamenti non di rado inattesi. Se uno di questi fenomeni è suscettibile di una descrizione matematica efficace e semplice, essa può essere considerata come un *modello matematico* di tutti gli altri fenomeni analoghi (od 'omologhi') […]
La scienza dei secoli passati era profondamente convinta che la matematica fosse capace di rappresentare l'autentica struttura dei fenomeni e che questa si riflettesse univocamente in un numero limitato di equazioni fondamentali. Tali sono l'equazione della meccanica di Newton, l'equazione del potenziale di Laplace, l'equazione delle vibrazioni, l'equazione del calore, le equazioni del campo elettromagnetico di Maxwell. Il grande fisico matematico ottocentesco J. Fourier (creatore della teoria matematica del calore) esprimeva tale credenza affermando che la natura e l'analisi matematica sono *equiestese*. Questa convinzione ha le sue radici nell'idea di Galileo secondo cui il grande libro della natura è stato scritto da Dio in caratteri matematici, e percorre tutta la storia della scienza moderna fino ad A. Einstein, il quale affermava che "abbiamo il diritto di essere convinti che la natura è la realizzazione di ciò che può essere immaginato di più semplice dal punto di vista matematico" (*Penser les mathématiques* 1982, p. 196). Se la natura è strutturata matematicamente, la matematica non è un mero strumento descrittivo ma permette di cogliere l'intima essenza dei fenomeni. La m. m. rappresenta quindi una prassi scientifica nuova, che emerge nella scienza del Novecento e anzi la caratterizza.»

it continued to be studied – as Clifford Truesdell pointed out[18] – by applied mathematicians or engineers working in applied mechanics.

In the 1930s, Kolmogorov significantly broadened his scholarly interests. From June 1930 to March 1931, he undertook his first academic journey abroad – visiting Göttingen, Munich, and Paris – together with Alexandrov.[19] In 1933, he published his landmark treatise on probability theory, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, in German. This work laid the axiomatic foundations of probability theory through the introduction of measure theory. Jan von Plato has noted that the intersection with physics played a role in the research that culminated in Kolmogorov's aforementioned treatise on probability.

«Two works precede *Grundbegriffe*'s axiomatization of measure theory [Kolmogorov, 1929, 1931]. In the latter, there was a physical motivation for constructing a theory of probability, namely the need to handle schemes of statistical physics in which time and state space are continuous.» (von Plato 2005, p. 962).

Moreover, Kolmogorov was among the participants in the lively seminar on the qualitative theory of differential equations, inspired by Birkhoff and led in Moscow from 1930 onward by his former professor Stepanov, together with Viktor Vladimirovich Nemytskii (1900-1967). Stepanov was closely connected both to the mathematical circle of Pavel Aleksandrov – which regarded Birkhoff's general theory of dynamical systems as an outstanding example of the application of set theory and topology, then new areas of mathematics, to mechanics and physics – and to the physical circle of Leonid Isaakovich Mandel'štam (1879-1944), who promoted a qualitative approach to nonlinear mechanics (Nemytskii 1957; Alexandrov et al. 1968).

As noted in Chapter 1, scholars in the USSR developed a strong interest in the work of Poincaré and Birkhoff, but focused largely on the study of dynamical systems in engineering contexts, as Simon Diner has pointed out. As for Kolmogorov, his 1985 note on his research in classical mechanics con-

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[18] See the *Introduction*.

[19] Kolmogorov had numerous mathematical contacts in Göttingen: "with Courant and his pupils in the field of limiting theorems, where diffusion processes proved to be limits for discrete random processes; with H. Weyl in intuitionistic logic; with E. Landau on questions in the theory of functions. He talked with Hilbert, had scientific contacts with E. Noether, H. Lewy, Orlicz, and many others" (Tikhomirov 1988, p. 10). On his contacts in Paris, see (Demidov 2009).

firms that he had followed the work of Birkhoff, Koopman, and von Neumann – described in Chapter 1 – suggesting, as Arnol'd's testimony implies, that he had been developing this line of research privately for years, without publishing anything, while continuing active work in other fields, such as information theory.

Kolmogorov did, however, publish a contribution on the dynamics of biological populations, following the path opened in the mid-1920s by the Italian scholar Vito Volterra, who used ordinary differential equations to study biological problems and aimed to construct a "mechanics of biological associations."

Why this *silent* work? Certainly, internal theoretical challenges may have contributed to the delay in overcoming the impasse left by Poincaré. But there are also contextual factors regarding the state of science under Stalinism that should be taken into account.

In recent years, several contributions have shed light on the circumstances of the so-called "Luzin affair"[20] – a revolt against Luzin's leadership within the Moscow mathematical community, which had been deeply affected by the actions of Kolman. The events took place in the summer of 1936, two years after the Academy of Sciences had relocated to Moscow and shortly before the mass repressions of 1937.

Moreover, Kolmogorov's interest in applying mathematics to biology would later bring him into conflict with Trofim Denisovich Lysenko (1898-1976), who, in 1940, firmly rejected any use of mathematics in the biological sciences.[21] Perhaps more significantly, working on the three-body problem may have been perceived as a risky endeavor. In October 1936, one of the leading figures in Soviet astronomy, Boris Vasilyevich Numerov (1891-1941?), a scholar active in celestial mechanics among other areas, was arrested – an early act in a broader campaign of repression targeting the USSR's astronomical community.

In the conclusion of this chapter, we now turn to this perhaps lesser-known episode in the history of Soviet science, which undoubtedly had an impact on the development of both mathematics and classical mechanics, and which merits further study.

[20] (Demidov, Lëvshin 2016), (Kutateladze 2013), (Levin 1990).

[21] See (Krementsov 1997); (Roll-Hansen 2005, 2008); (Joravsky 1960); (Graham 1993).

2.3.1 The Great Purge of Astronomers

The development of Soviet astronomy up to Stalin's death has been described by Efthymios Nicolaïdis in terms of three distinct periods. The first, lasting until the late 1920s, was marked by continuity in research but changes in organizational structure. The second period "sees the application of ideological concepts in astronomy and the official line of the superiority of Soviet astronomy over astronomy called bourgeois." The third period involved an unprecedented upheaval in the scientific personnel of Soviet astronomy (Nicolaïdis 1984, p. 6).

Astronomy proved to be a particularly sensitive field from two main perspectives:

«Until 1928, the change of social order in Russia affected astronomy only with respect to organizational matters: the new state organization was reflected in astronomical institutions by the creation of the Soviets of astronomers. At the ideological level the revolutionary state did not attempt to interfere in astronomy. Research and educational programs continued as before, except insofar as they concerned material problems.

After 1928 however, the Stalinist regime proclaimed a so called "Marxist" official ideological line concerning science. This ideological line became the official line of Soviet astronomy in 1931 Its principles were the following:

(1) *There are two sorts of astronomies Soviet and bourgeois*. This principle comes from the dogmatic principle that a capitalist regime restrains the scientific evolution while on the contrary, the construction of the socialist regime implies in addition the construction of a new, superior science. This principle of "two sciences" was the main Stalinist principle concerning all scientific fields. We will see that in astronomy, the application of this principle was to have terrible consequences for the leading Russian astronomers.

(2) Soviet astronomy must serve Soviet society more precisely *astronomy must serve ideology and the economy.*

But how could astronomy serve Stalinist ideology? [...] Astronomy was a scientific tool that would help to disprove what Stalinists called "religious myths". In a more specifically scientific field, soviet astronomy was ordered to fight against what was termed idealistic western cosmological theories, and especially against the theory of general relativity and the concept of a finite universe - because to put limits and an age to the universe would imply the Creation and so the existence of a God.

The relation between astronomy and the Soviet economy was a more complicated concept.
The general line that all activities in the USSR must serve the "building of socialism" implied that research in astronomy must also have industrial applications It was difficult to make applications concrete, and so the ideological line spoke about researches concerning Earth Sun relations and geodesy.» (Nicolaïdis 1990, pp. 346-347).

The ancient connection between the study of the stars and cosmological or theological questions, combined with the difficulty of identifying practical "applications" for astronomical research, led to sharp friction between the regime and the scholarly community. This tension culminated in one of the most dramatic episodes in the history of science under Stalinism: the purge that devastated the powerful group of astronomers working within the extensive network of observatories inherited from the Russian Empire.

The most prominent astronomers of the time – such as Aleksandr Aleksandrovich Ivanov (1867-1939), Boris Petrovich Gerasimovich (1889-1937), former and current directors of Pulkovo, and Boris Vasilyevich Numerov, director of the Leningrad Astronomical Institute – were not willing to conform to the new ideological line dictated by the regime. While not openly opposing it, they continued their research along the same lines as in previous decades. The only supporters of the new ideological direction were amateur astronomers or secondary figures.

In the summer of 1936, «Pravda» – the official press organ of the Communist Party of the Soviet Union from 1922 to 1991 – launched a brutal campaign against the Pulkovo Observatory. The attacks later extended to the Tashkent Observatory and the Leningrad Astronomical Institute. Gerasimovich, then director of Pulkovo, along with all its astronomers, was accused of subservience to foreign science (McCutcheon 1991; Eremeeva 1995).

Simultaneously with the decline in Pulkovo's reputation, a young student denounced Numerov. On 20 October 1936, he was arrested by the NKVD and, under torture, was forced to sign a document accusing many of his colleagues. Between late 1936 and the first half of 1937, approximately two dozen astronomers were arrested, leaving the Tashkent Observatory nearly deserted. Most of those arrested never returned. No further news of Numerov was ever received after 1941.

The final rehabilitations of those purged and who survived the repression date to 1956-57, leaving Soviet astronomy in a state of vulnerability for nearly twenty years. As Eremeeva, historian of astronomy at the Shternberg State Astronomical Institute (GAISh) in Moscow, has noted, it was only in the late 1960s that these tragic events were brought to light and the names of the astronomers who had fallen into disgrace could finally be acknowledged.

«The process of reclaiming the memory of the repressed astronomers "from Oblivion" was uneven and difficult. At first it was forbidden even to mention them in print. Indeed, it was the aim of the authorities to expunge not only their scientific work but their very names from human memory.
[...] Personal factors played an important role in the process of 'returning' the names of the repressed astronomers. Thus long before the rehabilitation process had begun, the names of the disgraced astronomers appeared in the 1948 jubilee compendium *30 years of astronomy in the USSR*. In an article about the development of fundamental astrometry, M. S. Zverev even mentioned the contributions of B. V. Numerov. S. A. Shorygin, who in his own time had suffered arrest, compiled the bibliography that included the works of B. P. Gerasimovich; the main text, however, included no mention of Gerasimovich.
Only in 1964 did the historian of astronomy Iu. G. Perel' dare to publish the first brief notes about Gerasimovich in the Soviet *Astronomicheskii kalendar*. Khrushchev's 1956 "secret speech" detailing Stalin's crimes to the 20th Congress of the Communist Party had by now become public knowledge, and the "thaw" of the 1960s had arrived. Thanks to the 'thaw' Perel was able to publish his article about Gerasimovich.» (Eremeeva 1995, p. 318).

Like many academics, Kolmogorov faced significant pressures during this period – both indirectly, as he witnessed the events unfolding in the field of celestial mechanics, and directly, when the mathematical community was shaken by the "Luzin affair" and when he became involved in genetics, publishing an article that contradicted the views of a student of Lysenko. Arnol'd later described Kolmogorov's state of mind during and after the Great Purges:

«Although Andrei Nikolaevich himself regarded the hopes that appeared in 1953 as the main stimulus for his work, he always spoke with gratitude about Stalin (following the old principle of saying only nice things about the dead): "First, he gave each ac-

ademician a quilt in the hard year of the war, and second, he pardoned my fight in the Academy of Sciences, saying, "such things happen also here". Andrei Nikolaevich also tried to speak kindly about Lysenko, who had fallen into disfavor, claiming that the latter had sincerely erred out of ignorance (while Lysenko was in power, the relation of Andrei Nikolaevich to this "champion in the struggle against chance in science" was quite different).
[...] "Some day I will explain everything to you," Andrei Nikolaevich used to tell me after having done something contrary to his principles. Seemingly, pressure was exerted on him by some evil genius whose influence was enormous (the role of the group transmitting the pressure was played by well-known mathematicians). He hardly lived to the times when it became possible to speak of these things, and, like almost all people of his generation who lived through the 1930's and 40's, he was afraid of "them" to his last day. One should not forget that for a professor of that time not to tell the proper authorities about seditious remarks made by an undergraduate or graduate student not infrequently meant being accused the next day of having sympathy with the seditious ideas (in a denouncement by the very same student-provocateur).» (Arnol'd 2000, p. 92).

3 Kolmogorov's research program for Hamiltonian dynamical systems: a look into the origins of KAM theory

> The statement of the problem of the motion of systems that are close to the systems of classical mechanics, including the problems of orbit evolution in the three-body problem, dates back to Newton. Laplace [2] stated explicitly the theorem on stability of the semimajor axes of Keplerian ellipses, which is a forerunner of Kolmogorov's theorem on preservation of tori, but proved it only in terms of approximate perturbation theory. On analyzing numerous attempts to justify and improve Laplace's argument, Poincaré [3] stated the problem in its modern form (to study the motion of a system whose Hamiltonian $W(p) + \theta S(q, p, \theta)$ is periodic in q) and called it the basic problem of dynamics (see [3], Chapter 1, §13). In the papers under consideration Kolmogorov solves this problem for the majority of initial conditions in the generic case ($det\partial^2 W/\partial p^2 \neq 0$).
>
> Vladimir Arnol'd in the introduction to Kolmogorov's *Selected papers*, vol. 1 (1985), section on classical mechanics.[1]

Four mathematicians from the USSR took part in the International Congress of Mathematicians held in Amsterdam in 1954 – the second congress held after the end of the Second World War, but the first in which Soviet mathematicians were allowed to participate.[2]

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[1] English translation in (Arnol'd 1991, p. 504). His references are to: I. Newton, *Philosophiae Naturalis Principia Mathematica*, London, 1686; P. S. de Laplace, *Traité de mécanique céleste*, vol. 1, Paris, 1799; and H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, Paris, 1892, chapter 1, §13.

[2] The 1936 Congress was suspended, and the series was reinstated only in 1950 (Cambridge, Massachusetts). On that occasion, however, the Soviet mathematical community did not participate. In the *Proceedings*, under the Secretary's Report section, the following note is included: "Shortly before the opening of the Congress, the following cable was received from the President of the Soviet Academy of Sciences:
The USSR Academy of Sciences appreciates having received a kind invitation for Soviet scientists to participate in the International Congress of Mathematicians to be held in Cambridge. Soviet mathematicians are very busy with their regular work, unable to attend the congress. I hope that the upcoming congress will be a significant event in mathematical science. Desire for success in congress activities. S. Vavilov, President,

During the closing plenary lecture on September 9, Kolmogorov presented to the international audience his ideas for a research program in the field of classical mechanics – specifically, "on the motion of systems close to the systems of classical mechanics, including the problems of orbit evolution in the three-body problem," as Arnol'd put it (in the note quoted above). He had been reflecting on this topic for a long time, as the testimonies reported in Chapter 2 suggest, and the ideas he presented encompassed those of two recent papers: "On dynamical systems with integral invariant on the torus" (Kolmogorov 1953) and "On the conservation of conditionally periodic motions under small variations of the Hamilton function" (Kolmogorov 1954), both published in Russian in the Proceedings of the Soviet Academy of Sciences («Doklady Akademii Nauk»), on November 13, 1953, and just nine days before the lecture, respectively.

The Russian text of the Amsterdam lecture was later published in 1957 in the *Proceedings of the International Congress of Mathematicians*, Amsterdam 1954 (Gerretsen, De Groot 1957). While the 1953 and 1954 papers are succinct and contain only a few references each, the written version of the Amsterdam lecture offers a broader overview that opens future avenues for research and includes 24 bibliographical references spanning the years 1917-1954.

On March 22, 1958, Kolmogorov gave a talk in Paris at the Seminar on Analytical Mechanics and Celestial Mechanics led by Maurice Janet (1888-1983). A French translation by Jean-Paul Benzécri (1932-2019) of the Russian text from the Amsterdam lecture (Kolmogorov 1957) was published in the seminar series (Kolmogorov 1958) (see Fig. 3.1). [3]

Ten years later, an English translation of the Amsterdam Proceedings text of Kolmogorov's 1957 lecture was published in the United States as an appendix to Ralph H. Abraham's (1936-2024) book *Foundations of Mechanics* (1967). The book was prepared with the assistance of Jerrold E. Marsden (1942-2010), based on his notes from a series of lectures given by Abraham in the Department of Physics at Princeton University, "aimed at recent

USSR Academy of Sciences." (Graves, Hille, Smith, Zariski, eds. 1955, p. 122).

[3] Kolmogorov's contribution was published in the first volume of the seminar series, corresponding to the academic year 1957-58. While some scholars had the opportunity to attend the Amsterdam lecture, in the absence of any audio or video recordings, this third paper is generally referred to as "the Amsterdam lecture." Further archival material concerning the preparation of the contribution for the Amsterdam *Proceedings* could enrich the analysis presented in this dissertation on the cultural origins of the theorem on the persistence of invariant tori.

Séminaire de
MÉCANIQUE ANALYTIQUE et de MÉCANIQUE CÉLESTE
dirigé par
Maurice JANET

1re année : 1957/58

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TABLE DES MATIÈRES

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Fig. 3.1. Table of contents from *Séminaire Janet. Mécanique analytique et mécanique céleste*, vol. 1 (1957-1958).

mathematical results in mechanics […] attended equally by mathematicians and physicists" (Abraham 1967, sixth printing 1987, p. xvii).

The Russian text of the Amsterdam ICM Proceedings, along with the two papers (Kolmogorov 1953, 1954), was later included in volume 1 of the *Selected Papers*, published in Moscow by «Nauka» in 1985 and edited by Vladimir Tikhomirov. The English translation for the 1991 edition was prepared by Vladimir M. Volosov.[4]

[4] The three English translations differ not in content, but in wording – even in the translation of paragraph titles. I have used the 1991 translation by Volosov.
Vladimir Markovich Volosov (born 1928) developed his scientific career in the field of non-linear mechanics, focusing on ordinary and partial differential equations, at *M. V. Lomonosov* Moscow State University. He graduated in 1950 from the Faculty of Physics, Department of Mathematics, earned his Ph.D. in 1956, and his D.Sc. in 1961. He later became a professor

In §3.1, I describe the content and rhetorical structure of the published text of Kolmogorov's Amsterdam lecture, which revolves around a central section (Section 3). This section features two questions in its title and can be understood as outlining a research program on Hamiltonian systems. The path is traced through a rich bibliography that, on the one hand, anchors Kolmogorov in past research, while on the other, builds upon conjectures pointing toward future developments.

The core of Kolmogorov's research program consists of two theorems: one on the persistence of invariant tori, and the other on the Lebesgue measure of persistent invariant tori – both concerning analytic, nearly-integrable Hamiltonian systems (with a nondegenerate integrable Hamiltonian and bounded phase space). These theorems were first stated in Kolmogorov (1954) and later presented in the Amsterdam lecture. They will be the primary focus of §3.2.

In this chapter, I also analyze the Diophantine condition, which plays a key role in the proof of Theorem 1, comparing its application to that found in a 1942 article by the German mathematician Carl Ludwig Siegel (1896-1981). Siegel, together with Jürgen Moser, authored *Lectures on Celestial Mechanics* (1956), based on Moser's notes from a series of Siegel's lectures delivered in 1951-52.

Finally, in §3.3, I address Kolmogorov's broader research program, aimed at resolving longstanding problems in celestial mechanics and opening new avenues for the study of Hamiltonian dynamical systems.

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and a member of the *P. P. Shirshov* Institute of Oceanology of the Academy of Sciences. Volosov had previously translated a 1980 textbook by Viacheslav M. Starzhinskii, which was published in English by MIR (Moscow) under the title *Advanced Course of Theoretical Mechanics for Engineering Studies* (1982). See: <www.mathnet.ru/eng/person12469>.

3.1 “The many-sided interrelations with the most varied branches of mathematics”: A metric and spectral approach to the problems of classical mechanics

> I will consider my objective accomplished if I have managed to convince the audience that, in spite of the great difficulties and the limited nature of the results already obtained, the problem I have posed of using general notions of modern ergodic theory for qualitatively analyzing motion in analytic and, particularly, canonical dynamical systems deserves great attention of scientists capable of comprehending the many-sided interrelations with the most varied branches of mathematics revealed here.
>
> (Kolmogorov 1957, pp. 372-373)

With the words quoted above, Kolmogorov concluded the text of his lecture at the ICM in Amsterdam. The “problem” or issue at stake was presented as a challenge to scholars willing to explore the internal connections between different areas of mathematics – including classical mechanics – rather than remain confined to specialized research fields.

In Table 3.2, the paragraphs of the published text (Kolmogorov 1957) are listed to guide the description of how Kolmogorov developed his subject. He captured the attention of his audience and readers through examples and direct questions, inviting them on a journey through dynamics, and approaching evolution problems from a metrical and spectral point of view, as suggested by the “modern means” of twentieth-century mathematics. While outlining his research program, he also advanced several conjectures, which contribute to the overall sense of excitement in the presentation.

In fact, beyond presenting two theorems concerning Hamiltonian systems, the published text of the lecture reveals that Kolmogorov had a broader project in mind. His program envisaged a wide-ranging study of dynamical systems – not focused on a specific case or a single dynamical system describing a particular real event, but rather aimed at establishing a general method. This method would allow one to determine which properties can be considered “general” or “exceptional” (in the sense of measure theory), both for the function defining the system and for the orbits it describes.

Let us now consider the approach he adopts in the Introduction:

The general theory of dynamical systems and classical mechanics

Introduction

§1. Analytic dynamical systems and their stable properties

§2. Dynamical systems on a two-dimensional torus and some canonical systems with two degrees of freedom

§3. Are dynamical systems on compact manifolds "in general" transitive, and should we regard the continuous spectrum as the "general" case and the discrete spectrum as an "exceptional" case?

§4. Some remarks on the non-compact case

§5. Transitive measures, spectra, and eigenfunctions of analytic systems

Conclusion

Table 3.2. Kolmogorov's 1954 Amsterdam lecture: an overview of its contents through the paragraphs of the published text (Kolmogorov 1957).

«My aim is to elucidate ways of applying basic concepts and results in the modern general metrical and spectral theory of dynamical systems to the study of conservative dynamical systems in classical mechanics.
[...] the Poincaré-Carathéodory[5] recurrence theorem initiated the "metrical" theory of dynamical systems in the sense of the study of properties of motions holding for "almost all" initial states of the system. This gave rise to the "ergodic theory", which was generalized in different ways and became an independent centre of attraction and a point of interlacing for methods and problems of various most recent branches of mathematics (abstract measure theory, the theory of groups of linear operators in Hilbert and other infinite-dimensional spaces, the theory of random processes, etc.).

[5] Constantin Carathéodory (1873-1950) was a mathematician of Turkish-Greek origin, best known for his work in mathematical analysis. See (Georgiadou 2004) and (Phili 2002).

[...] For conservative systems, the metrical approach is of basic importance making it possible to study properties of a major part of motions. For this purpose, contemporary general ergodic theory has elaborated a system of notions whose conception is highly convincing from the viewpoint of physics.» (Kolmogorov 1957, pp. 354-355).

He begins to illustrate the *modus operandi* he intends to adopt in his research through a specific problem: a motion defined on an *s*-dimensional manifold, $\Omega_{2s,}$ by a Hamiltonian system $H\,(q_1,..., q_s\,, p_1\,, . . . , p_s)$, where $(q_1, . . . , q_s)$ are position coordinates and $(p_1, . . . , p_s)$ are momenta.

Assuming that the motion admits *k* prime integrals[6]:

$$I_1 = C_1, . . . , I_k = C_k\,,$$

these integrals, being constant functions along the motions, reduce the number of degrees of freedom of the system from 2*s* to 2*s* - *k*, and define within the phase space an analytic manifold, denoted $M_{2s\text{-}k}$.
An invariant density can be defined on this manifold, which Kolmogorov denotes by $M(x)$. This function is key to applying the methods of measure theory in dynamical systems to motions on $M_{2s\text{-}k}$:

«It is reasonable to resort to these more modern means when, apart from the integrals $I_1 = C_1, . . . , I_k = C_k$, there are no single-valued analytic first integrals independent of the former or when their determination encounters severe difficulties and other classical methods for completing the integration of the system also prove inapplicable. In such cases it is necessary to use a qualitative approach in order to find out whether the motion on $M_{k\text{-}2s}$ is transitive (that is, whether almost the entire manifold $M_{k\text{-}2s}$ consists of a single ergodic set) and then, in the transitive case, to determine the nature of the spectrum or, in the absence of transitivity, to study, to within a set of measure zero (or at least to within a set of small measure), the decomposition of $M_{k\text{-}2s}$ into ergodic sets and the nature of the spectrum on these ergodic sets.
There are only two specific problems of classical mechanics known to me where this programme has been realized to a certain degree.
[...] However, I believe that the time has now come when considerably more rapid

[6] Kolmogorov's prime integrals correspond to Poincaré's invariant integrals (see §1.1.2).

progress can be made.» (Kolmogorov 1957, p. 357).

His intent, as described in the Introduction, is put into action in the five paragraphs that structure his argument and include references to the papers and books listed in the bibliography. The published text of his Amsterdam lecture (Kolmogorov 1957) contains a bibliography with 24 references to papers and monographs by 23 authors, published between 1917 and 1954, from five countries outside the USSR, including the United States. The list also includes Kolmogorov's own papers from 1953 and 1954.

The oldest reference is Émile Borel's "Leçons sur les fonctions monogènes uniformes d'une variable complexe" (1917), and the most recent is the 1954 paper by the Soviet mathematician Mstislav Igorevich Grabar (1925-2006), "On Strongly Ergodic Dynamical Systems". Table 1 presents the group of scholars cited in the bibliography of Kolmogorov's 1957 text.[7] The oldest among them is Ludwig Becker, while the youngest are two Soviet mathematicians born in the 1920s: Mstislav Grabar and Kirill Aleksandrovich Sitnikov.

[7] The name of G. K. Badalyan also appears in the "long list", but no biographical information could be found.

USSR	Finland	Germany	France	UK	USA
				Ludwig Becker (1860-1948)[1]	
	Hjalmar Tallqvist (1870–1958)		Borel Emile (1871-1956)		
Nikolay N. Krylov (1879-1955)[2]					George D. Birkhoff (1884-1944)
Vyacheslav V. Stepanov (1889-1950) Otto Y. Schmidt (1891-1956) Aleksandr Y. Khinchin (1894-1959)					
Viktor V. Nemytskii (1900-1967) Andrej A. Markov (1903-1979) Lev D. Landau (1908-1968) Nikolay N. Bogoliubov (1909-1992)		Eberhard Hopf (1902-1973)			John von Neumann (1903-1957)
Leonid V. Kantorovich (1912-1986) Izrail' M. Gelfand (1913-2009) Sergei V. Fomin (1917-1975)					Shizuo Kakutani (1911-2004)[3]
Mstislav I. Grabar (1925-2006) Kirill A. Sitnikov (b. 1926)					

[1] German-born, he worked in the United Kingdom from 1888 onward.
[2] In Kyiv, the current capital of Ukraine.
[3] Japanese-born, he worked between the United States and Japan, primarily at Princeton and Yale Universities.

Table 3.3. Scientists cited in Kolmogorov's works (1953, 1954, 1957, 1985) and in Arnol'd (2000), ordered by date of birth and nationality.

Section 1 is devoted to concepts and notation that will be used in the following central four paragraphs:

1. A *dynamical system of classical mechanics* – which, he underlines, is "a special case of analytic dynamical system with an integral invariant" – is defined by the differential equation

$$\frac{dx_\alpha}{dt} = F_\alpha(x_1, \ldots, x_n)$$

on a manifold Ω_n with $\alpha = 1, \ldots, n$

2. The *invariant measure* is defined by the integral

$$m(A) = \int_A M(x) dx_1 \ldots dx_n$$

where M(x) is the invariant density defined above.

3. A *canonical system* is defined as a dynamical system represented by a Hamiltonian function in the variables $(q_1, \ldots, q_s)$ and $(p_1, \ldots, p_s)$ on a manifold such that

$$\frac{dq_\alpha}{dt} = \frac{\partial H}{\partial p_\alpha}, \qquad \frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial q_\alpha}$$

and with the invariant density equal to one:

$$M(q, p) = 1.$$

Having introduced the mathematical objects, he turns back to clarifying the *modus operandi*:

«Particular attention will be paid to finding which of the properties of dynamical systems are "typical" for "arbitrary" $F\alpha$ and M (or an "arbitrary" function $H(q, p)$ in the case of canonical systems) and which of them can manifest themselves only by way of an "exception". However, this is quite an intricate problem. The approach from the standpoint of the category of corresponding sets in the spaces of systems of functions $\{F\alpha, M\}$ (or functions H), despite the well-known achievements in this direction obtained in the general theory of abstract dynamical systems, is of interest

rather as a means for proving existence than as a direct way for solving actual problems set by researchers in physics and mechanics, however stylized and idealized their statement may be. By contrast, *the approach from the standpoint of measure theory* appears to be quite reasonable and natural as viewed from physics (for instance, as it was set forth forcibly by von Neumann [1]), but its application is hampered by the absence of a natural measure in function spaces.
We will follow two routes. First, to obtain positive results establishing that a certain type of dynamical systems should be recognized as being essential, not "exceptional", and from any reasonable point of view, should not be "neglected" (in the way that sets of measure zero are neglected), we will use the notion of stability in the sense of preservation of a certain type of behaviour of a dynamical system under small variation of the functions F_α and M or of the function H . From this standpoint, any type of behaviour of a dynamical system for which there exists at least one example of its stable realization should be recognized as being important and not negligible.» (Kolmogorov 1957, pp. 358-359, my emphasis).

Note the use of inverted commas for terms such as "exceptional," "neglect," or "typical."

In Section 2, *Dynamical Systems on a Two-Dimensional Torus and Some Canonical Systems with Two Degrees of Freedom*, Kolmogorov begins to demonstrate his approach in practice by considering a dynamical system defined on a two-dimensional manifold – specifically, a two-dimensional torus, T_2. He justifies the choice of this example as follows:

«Therefore the real significance for classical mechanics of the above analysis of dynamical systems on T_2 depends on whether there are sufficiently important examples of canonical systems with two degrees of freedom, not integrable by classical methods.» (Kolmogorov, 1957, p. 363).

It is in this section that we find the most references to the two previously published papers – particularly an application, to the specific case under examination, of the theorem on the persistence of invariant tori, which had been published in general form just nine days earlier.[8]

Following an example, the discourse gains momentum in Section 3, marked by the formulation of two questions that make up its title:

[8] For further details, see §3.2 on the theorem concerning the persistence of invariant tori.

1) *Are dynamical systems on compact manifolds "in general" transitive?*
2) *Should we regard the continuous spectrum as the "general" case and the discrete spectrum in an "exceptional" case?*

Both questions – and the central Section 3 – serve as the fulcrum of the lecture: they express Kolmogorov's aim to clarify which properties of a general dynamical system can be considered general or exceptional, thereby revealing the core of his research program.

It is here that the connection with ergodic theory emerges, particularly through the concept of transitivity.[9] The negative answer to both questions sets a clear boundary to the widespread conjecture that the ergodic hypothesis applies universally to all dynamical systems.

«This contradicted claims which one could often see in the physical literature according to which any typical Hamiltonian system with interaction should be ergodic.» (Sinai 1989, p. 838)[10]

In (Arnol'd 1991), the author emphasizes that some conjectures proposed by Kolmogorov in this section – concerning particular systems in which cases of mixing on tori would occur (as discussed in Kolmogorov 1953) – were later proved in two papers by Yakov Sinai and Dmitri Victorovich Anosov in the 1960s:

«Systems with stable transitivity and mixing on the energy level surfaces which Kolmogorov discusses at the end of §3 of the lecture at the Amsterdam Congress (paper No. 53) actually exist. Sinai and Anosov proved that geodesic flows on compact manifolds of negative curvature (along each two-dimensional direction) possess these properties [46-48][11]. Moreover, these properties are preserved under small perturbations not only in the class of Hamiltonian systems but also in the class of general dynamical systems.» (Arnol'd 1991, p. 510).

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[9] See §1.2 above.

[10] One paper that aimed to demonstrate the opposite is Enrico Fermi's "Dimostrazione che in generale un sistema meccanico è quasi ergodico" (Proof that in general a mechanical system is quasi-ergodic), published in the journal «Nuovo Cimento» (Fermi 1923a).

[11] He refers to (Sinai 1966) and (Anosov 1967).

Section 4 is devoted to the discussion of a system defined on a non-compact manifold. The goal is to extend the results obtained in the previous sections to the study of trajectories that escape from every compact region Ω as $t \to \infty$ and $t \to -\infty$. Kolmogorov emphasizes the need to construct an "individual ergodic theory" to address this case.[12]

To this end, he employs the same methods used by Krylov and Bogolyubov in their 1937 paper (Krylov, Bogolyubov 1937), already discussed in Chapter 1, §1.2.3, although their work was in the context of nonlinear mechanics with compact manifolds. Here, the problem of ultimate motions in the three-body problem is addressed – an issue later studied in depth by Kolmogorov's students, Kirill Aleksandrovich Sitnikov and Vladimir M. Alekseev (1932-1980).

Finally, in the concluding Section 5, Kolmogorov turns to aspects closely related to the overseas works of Birkhoff, Koopman, and von Neumann, particularly concerning spectra and transitive measures:

«The spectral properties of transitive measures in analytic systems have not been studied enough.» (Kolmogorov 1957, p. 371).

In particular, he puts forward two conjectures:

1. the stability of a dynamical system with continuous spectrum, and
2. that a countable discrete spectrum is "typical" in analytic dynamical systems.

«It is not impossible that only these cases (a discrete spectrum with a finite number of independent frequencies and a countably-multiple Lebesgue spectrum) are admissible for analytic transitive measures or that, in a sense, only they alone are general typical cases.» (*Ibidem*)

It is again Sinai and Asonov who proved, in 1966 and 1967 respectively,[13] the first of Kolmogorov's conjectures presented in this section.[14]

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[12] In the compact case discussed in Sections 2 and 3, the "classical" results of ergodic theory can be applied, due to the availability of an invariant measure (see §1.2.3).

[13] (Sinai 1966) and (Asonov 1967).

[14] "Kolmogorov's conjecture (§5 of paper No. 53) on the stability of a continuous (more precisely, countably-multiple Lebesgue) spectrum was proved by Sinai and Anosov [46, 47].

In the last two sections of the published text of the Amsterdam lecture all the influences that we have explored in Chapter 1 emerge, that set up the cultural landscape as background of Kolmogorov's research programme.[15] Some further discussion is needed here, and I present it in §3.3. But let's turn now to the crucial theorem on the persistence of invariant tori, that he had already worked out before presenting his closing lecture at Amsterdam.

3.2 Kolmogorov flips his cards: the paper "The preservation of conditionally periodic motions under small variations of the Hamiltonian function" (1954)

The paper "The preservation of conditionally periodic motions under small", written in Russian and published in «Doklady Akademii Nauk» in August 1954, is stylistically quite different from the published text of the Amsterdam lecture. Just four pages[16] it focuses on the formulation of two theorems (unnamed, numbered Theorem 1 and Theorem 2) and provides a demonstration of the first, followed by a brief concluding remark.

Theorem 1 concerns the persistence of invariant tori for analytic, nearly-integrable Hamiltonian systems[17]. Theorem 2 addresses the Lebesgue measure of persistent invariant tori in such systems, assuming a nondegenerate integrable Hamiltonian and a bounded phase space.

Thus far the conjecture that a discrete spectrum with a finite number of independent frequencies (not exceeding the phase space dimension) and a countably-multiple Lebesgue spectrum is typical has not been refuted for analytic systems." (Arnol'd 1991, p. 513)

[15] On its impact after 1954, Arnol'd wrote in 1991: "Kolmogorov's classical papers Nos. 52 [see below, §3.2] and 53 [the Amsterdam lecture] produced a very strong effect on the subsequent development of the theory of dynamical systems, and at present there are dozens of books developing or presenting the material of these papers." (Arnol'd 1991, p. 504)

[16] Six pages in the version published in Volume 1 of *Kolmogorov's Selected Works* (English edition).

[17] I adopt the terminology used by Arnol'd: "Kolmogorov's 1954 theorem on the persistence of invariant tori under a small analytical perturbation of a fully integrable Hamiltonian system," and "he [Kolmogorov] arrived at his theorem of 1954 on the persistence of invariant tori" (Arnol'd 1997, pp. 742-743). The result is sometimes referred to as the "KAM theorem" (see below, § 3.3.1). Naturally, Kolmogorov's original notation differs significantly from those currently in use; see (Chierchia 2008).

The fate of these theorems is rather peculiar. Theorem 1, due to the nature of Kolmogorov's 1954 proof, is often conflated with subsequent contributions from the 1960s by his former student Arnol'd (Arnol'd 1963) and the German mathematician Jürgen Moser (1928-1999) (Moser 1962). This has led to the perception that the latter two completed the actual proof. In fact, Kolmogorov's Theorem 1 has been effectively "merged" with the contributions of Arnol'd and Moser, with all three theorems now commonly referred to as the "KAM Theorem." Theorem 2, by contrast, has been almost entirely forgotten. It appears at the end of the 1954 paper, without proof. Arnol'd, in (Arnol'd 1963), merged the two theorems into a single statement (Chierchia, Fascitiello 2024).

Theorem 1, on the persistence of invariant tori, was considered and reformulated in the published version of the Amsterdam lecture – first in Section 2, using the example of a Hamiltonian system defined on a two-dimensional torus, and then in Section 3, where it is stated in the general case of a 2s-dimensional manifold.

3.2.1 The statement and meaning of the theorem on the persistence of invariant tori in nearly integrable Hamiltonian systems

After a few introductory lines in which the mathematical objects considered in the paper are presented, the statement of the theorem on the persistence of invariant tori (Theorem 1 in the paper) follows. It is quite long and detailed. This is followed by a brief remark from the author on the broader significance of the theorem for (classical) mechanics, and then by an extended discussion of its proof (approximately 2.5 pages).

This discussion does not constitute a full formal exposition of the deductive proof; rather, the author outlines the development of his argument point by point, without dwelling on individual logical or mathematical steps.

We can divide the section devoted to the proof into three main phases:

1. A brief initial explanation of the method used;
2. A more technical central section, in which some mathematical steps in the proof are explained;
3. A final part, introduced by the words "It is easy to see that [...]" (Kolmogorov 1954, p. 352), where mathematical rigor gives way to a broader,

more conversational explanation.

We will consider additional aspects of Kolmogorov's discussion of the proof in §3.3.1.

Here, however, let us examine the original 1954 formulation of the theorem and its significance, which is connected both to problems in classical mechanics – particularly celestial mechanics – and to ergodic theory.

Adopting the notation and wording used by Kolmogorov, we consider a region $G \subset \Omega_{2s}$, the phase space represented as the product of an s-dimensional torus T and a region S in an s-dimensional Euclidean space.

In this setup, the points of the torus are characterized by periodic functions of period 2π, $q_1, \ldots, q_s$, and the coordinates of a point p in the region S are denoted by the vector $p_1, \ldots, p_s$.

So, we consider a Hamiltonian H in G having the canonical form

$$\frac{dq_\alpha}{dt} = \frac{\partial H}{\partial p_\alpha}, \qquad \frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial q_\alpha}$$

and suppose that:

- H also depends on a parameter θ (a perturbative parameter), where $\theta \in (-c\,;\,c)$ and is independent of time;
- H is analityc in the variables (q, p, θ).

From this point on, we will consider the Hamiltonian function H defined on G, with $\theta = 0$ taking the form:

$$H(q,p,0) = m + \sum_\alpha \lambda_\alpha p_\alpha + \frac{1}{2}\sum_{\alpha\beta} \Phi_{\alpha\beta}(q) p_\alpha p_\beta + O(|p^3|),$$

where, m is a real constant and represents the constant energy of the system, $\alpha, \beta \in (1, \ldots, s)$ are integers, $(\lambda_1, \ldots, \lambda_s)$ are the frequencies of motions, the sum coincides with the scalar product between the vector of the frequencies $(\lambda_1, \ldots, \lambda_s)$ and the vector $(p_1, \ldots, p_s)$ of the coordinates of a point belonging to S.

The meaning of $\Phi_{\alpha\beta}$ will be clearer in the statement of the theorem.

Finally, we will denote with

$$(x, y) = \sum_\alpha x_\alpha y_\alpha, \text{ and } |x| = +\sqrt{(x,x)}$$

and with T_c an s-dimensional torus in the region G, formed by the set of points (q, p) with $p = c$ constant.

We will assume that S contains the point $p = 0$, that is, $T_0 \subseteq S$.

Kolmogorov's theorem on the persistence of invariant tori

Theorem 1[a] Let

$$H(q,p,0) = m + \sum_{\alpha} \lambda_\alpha p_\alpha + \frac{1}{2} \sum_{\alpha\beta} \Phi_{\alpha\beta}(q) p_\alpha p_\beta + O(|p^3|),$$

where m and λ_α are constants, and let the inequality

$$|(n,\lambda)| \geq \frac{c}{n^\eta}$$

be fulfilled for a certain choice of the constants $c > 0$ and $\eta > 0$ and all integral vectors n. Morover, let the determinant formed from the average values

$$\Phi_{\alpha\beta}(0) = \frac{1}{2\pi^s} \int_0^{2\pi} \int_0^{2\pi} \Phi_{\alpha\beta}(q)\, dq_\alpha\, dq_\beta$$

of the function

$$\Phi_{\alpha\beta}(q) = \frac{\partial^2 H}{\partial p_\alpha \partial p_\beta}(q,0,0)$$

be non zero:

$$|\Phi_{\alpha\beta}(0)| \neq 0.$$

Then there exists analytic functions $F_\alpha(Q, P, \theta)$ and $G_\alpha(Q, P, \theta)$ defined for all sufficiently small θ and all point (Q, P) belonging to a neighbourhood V of the set T_0 that determine a contact transformation

$$q_\alpha = Q_\alpha + \theta\, F_\alpha(Q, P, \theta), \qquad p_\alpha = P_\alpha + \theta\, G_\alpha(Q, P, \theta)$$

of V into $V \subseteq G$ reducing H to the form

$$|\Phi_{\alpha\beta}(0)| \neq 0.$$

(M (θ) does not depend on Q or P).

[a] Original version in Kolmogorov (1957), pp. 349-350.

Kolmogorov himself provides an explanation of the meaning of such theorem on dynamical systems in the case of mechanics:

«The significance of Theorem 1 in mechanics can easily be understood. It shows that, under conditions (2) and (3), an s-parameter family of conditionally periodic motions

$$q_\alpha = \lambda_\alpha t + q_\alpha^{(0)}, \qquad p_\alpha = 0,$$

existing at $\theta = 0$ cannot disappear under a small variation of the Hamilton function H ; namely, the variation results only in a displacement of the s-dimensional torus T_0, along the trajectories of the motions: it is transformed into a torus $P = 0$, which is filled with trajectories of conditionally periodic motions with the same frequencies $\lambda_1, \ldots, \lambda_s$.» (Kolmogorov 1957, p. 350).

The condition is $|(n, \lambda)| \geq \frac{c}{n^\eta}$ what is now known as the *Diophantine condition*. It is a condition that can be imposed on an irrational real number λ. It can be shown that this condition holds for almost all irrational real numbers, except for a set of Lebesgue measure zero (I will examine this in greater detail in §3.2.2).

Thus, Kolmogorov stated that for most initial frequencies – i.e. for all those satisfying the Diophantine – the motions of the perturbed Hamiltonian system remain quasi-periodic. The tori that foliate the phase space in the unperturbed Hamiltonian system are not destroyed by the perturbation but are instead transformed into nearby invariant tori, on which the motions remain quasi-periodic with the same frequencies $\lambda_1, \ldots, \lambda_s$.

Sinai emphasized the close connection between this result and Poincaré's earlier work:

«Like Poincaré, Kolmogorov considered small perturbations of integrable systems and proved that most invariant tori in the measure-theoretic sense are preserved under small perturbations.» (Sinai 1989, p. 838).

Indeed, Kolmogorov's theorem represents an important contribution to the "general problem of dynamics," originally formulated by Poincaré (see above, §1.2.1), and to its subsequent developments, which remained unresolved for more than fifty years. Under the conditions imposed by Kolmogorov, and for sufficiently small perturbations θ, nearly integrable perturbed Hamiltonian systems are stable for $s \leq 2$, and for $s \geq 3$, the majority of initial conditions yield solutions that remain stable for all time.

How and where does Kolmogorov place Theorem 1, on the persistence of invariant tori, in the published version of his Amsterdam lecture? First, in Section 2, he considers a particular case of the theorem: a perturbed Hamiltonian system with two degrees of freedom on a two-dimensional torus.

As shown above (§3.1), Section 3 addresses the core of Kolmogorov's research program: the attempt to identify general or exceptional properties of dynamical systems. He immediately clarifies that the two questions posed in the section's title are related to issues in ergodic theory (see §1.2.3).[18] Kolmogorov shows that the answers to both questions are negative for analytic canonical systems:[19] canonical Hamiltonian systems are, in general, neither transitive nor do they possess a continuous spectrum.

Kolmogorov outlines only the essential points of the proofs of these two claims, along with the differences between his results and earlier approaches in perturbation theory.

«The method of proof consists in studying the behaviour of the original tori T^2_c with frequencies $\lambda_\alpha(c)$ satisfying condition[20] under variation of θ[21] and establishing that for sufficiently small ϵ each of the tori is not destroyed and is merely displaced in Ω with preservation of trajectories of conditionally periodic motions with constant frequencies λ_α on it.

Probably *many of you will already have guessed that*, in essence, what we are talking about is a certain modification of the idea of the possibility of avoiding the appearance

[18] Recall from §1.2.3 that if T is a measure-preserving transformation on a space X, then T is ergodic (or transitive) if and only if it has only trivial invariant sets – that is, if and only if $m(E) = 0$ or $m(X - E) = 0$ whenever E is a measurable set invariant under T.

[19] "In application to analytical canonical systems, the answers to both questions are negative." (Kolmogorov 1957, p. 365).

[20] Condition 2 corrisponds to a Diophantine condition (see above).

[21] That is, the small perturbation of the system.

of abnormal "small divisors" when calculating disturbed orbits, which has been extensively discussed in the literature on celestial mechanics. However, in contrast to ordinary perturbation theory, we obtain exact results instead of the conclusion that the series of some approximation of finite order (relative to θ) are convergent. This is achieved because instead of calculating the disturbed motion for fixed initial conditions, we change the initial conditions themselves so that, with varying θ, we always deal with motions having normal frequencies λ_α (in the sense of condition (2)).» (Kolmogorov 1957), p. 364, my emphasis).

So, he observes:

«In application to analytic canonical systems, the answers to both questions are negative, since the theorem on the stability of the decomposition into tori which we stated for systems with two degrees of freedom remains valid for any number of degrees of freedom as well.
[...] Thus, under small variations of H the dynamical system remains non-transitive and the region G continues to be decomposable, to within a residual set of small measure, into ergodic sets with discrete spectra (of the indicated specific nature).» (Kolmogorov 1957, pp. 365-366).

Thus, canonical Hamiltonian systems are, in general, neither transitive nor do they possess a continuous spectrum.

Theorem 1, as obtained by Kolmogorov, therefore inevitably raises questions about the general validity of the ergodic hypothesis.

However, as long as the number of degrees of freedom is finite, the primary applications of the ergodic hypothesis remain valid. By contrast, in contexts such as statistical mechanics – where the number of degrees of freedom is very large or tends to infinity – Theorem 1, in its stated form, does not generally apply.

Furthermore, Kolmogorov notes:

«No similar results regarding the stability of a certain general type of behavior of non-canonical dynamical systems with an integral invariant and a compact phase space Ω^n are known to me.» (Kolmogorov 1957, p. 367).

3.2.2 The diophantine condition: from Carl Ludwig Siegel (1896-1981) to Kolmogorov

In 1942, Carl Ludwig Siegel (1896-1981) published the article "Iteration of analytic functions in the «Annals of Mathematics»". In his attempt to solve convergence problems in a Fourier series, he employed the same Diophantine condition on irrational real numbers that Kolmogorov would adopt twelve years later (see above, §3.2.1). While the contexts differ, the question has naturally arisen: was Kolmogorov aware of Siegel's work? Identifying this condition is crucial, as it constitutes one of the key assumptions in Kolmogorov's Theorem 1.

Siegel's work on analytic functions

Siegel, a German mathematician of Kolmogorov's generation, was a specialist in number theory with an early interest in astronomy. He began his mathematical education in Berlin and, after the Great War, defended his doctoral thesis in Göttingen in 1920. In 1922, he was appointed lecturer at the Johann Wolfgang Goethe University in Frankfurt, where he remained until 1938, when he accepted a professorship in Göttingen. However, in the spring of 1940, he left Germany[22] and became a fellow at the Institute for Advanced Study in Princeton, where he remained until obtaining a permanent position in 1946. In 1951, he returned permanently to Göttingen.

Already a leading figure in the development of number theory, Siegel turned in the 1940s to the theory of analytic functions. His dual interest in number theory and mathematical analysis is attested by Jean Dieudonné (1906-1992):

«La renommée universelle de Siegel est surtout due à ses travaux de Théorie des nombres, où il s'inscrit dans la grande lignée qui commence avec Fermat et se poursuit avec Euler, Lagrange, Gauss et les brillantes écoles allemande et française du XIX siècle. Mais on lui doit aussi d'importants résultats en théorie des fonctions de plusieurs variables complexes et en mécanique céleste; il est d'ailleurs frappant que tous ses mémoires de Théorie des nombres reposent sur un maniement de

[22] Siegel's "escape" from Germany is recounted by himself in the text of his Address given on June 13, 1964, in the Mathematics Seminar of the University of Frankfurt on the Occasion of the 50th Anniversary of the Johann Wolfgang Goethe University Frankfurt (Siegel 1979).

l'Analyse mathématique d'une profondeur et d'une virtuosité incomparables.» (Dieudonné 1983, p. 63).

Siegel gave lectures on celestial mechanics in Frankfurt and Main, Baltimore, Princeton, and Göttingen. In Göttingen, during the winter semester of 1951-52, he delivered a lecture series which, based on notes taken by the student Morse, formed the basis for the first edition of *Lectures on Celestial Mechanics*, published in 1956 (see Siegel and Moser 1995/1971, revised edition).

Two papers – Siegel 1941 and 1942 – focused on a classical linearization problem related to celestial mechanics. In the first, Siegel demonstrated that every convergent integral (solution) of a given canonical system can be expressed as a power series in a single variable. However, it is Siegel himself who warns the reader:

«This elegant method of solution has also been generalized to the case of a function H which contains explicitly the variable *t*, in periodical form, and is closely related to the important researches of Delaunay, Hill[23] and Poincaré in celestial mechanics. However, there is a serious objection: The question of convergence has never been settled.» (Siegel 1941, p. 807).

In the 1942 paper "Iteration of analytic functions", already mentioned above, Siegel goes one step further: the analytic power series

$$f(z) = \sum_{k=1}^{\infty} a_k z^k$$

with the assumption that a_1 is a number such that $|a_1| = 1$ and $a_n \neq 1$ for $n = 1, 2. \ldots$ and

$$log|a_n - 1| = O(log\ n)$$

is convergent.

It is then Siegel himself who states that the hypothesis above on a_1 is equivalent to stating that, writing a_1 in the exponential form $a_1 = e^{2\pi\omega}$ then

[23] Charles-Eugène Delaunay (1816-1872), French astronomer and mathematician; and George William Hill (1838-1914), American astronomer and mathematician.

$$|\omega - \frac{m}{n}| > \lambda n^{-\mu}$$

for arbitrary integers m and n, $n \geq 1$, where λ and μ are positive numbers, depending only upon ω. This expression is precisely the Diophantine condition that Kolmogorov imposed on the frequency of motion to obviate the problem of small divisors which would have interfered with the convergence of the series.

Did Kolmogorov know about Siegel's work?

Although the demonstrative techniques used by Siegel in 1942 and by Kolmogorov in 1954 are entirely different, the coincidence of their use of the same Diophantine condition has inevitably raised the question posed in the title of this subsection – and the historical conjecture that Kolmogorov was indeed familiar with the work of his German colleague (see Dumas 2014, pp. 15, 35, 64, 81; Goldstein 1980, p. 530).

Let us quote some excerpts from Dumas (2014):

«Occasionally, disagreement erupts over how much Kolmogorov proved in 1954. [...] Still others think that C.L. Siegel's name should be attached to the theorem.» (Dumas 2014, p. 15).

And again:

«Kolmogorov (may have) adapted this step from Siegel's work, as described above.» (Dumas 2014, p. 64).

He further clarifies his reason for writing "(may have)":

«In describing the first solutions of small divisor problems, many references say something like "Kolmogorov adapted Siegel's techniques," as I do here. However, in the sequel I'll qualify this with 'perhaps,' because, while there is no doubt that Siegel's work on small divisors preceded Kolmogorov's by a dozen years, there does not seem to be direct evidence that Kolmogorov knew about Siegel's work.»

And to support his statement, he refers to an autobiographical paper by Arnol'd:

«I started inquiring whether somebody had examined all these questions between A. Denjoy's work of 1932[24] and my work of 1958. Among others, I found C. L. Siegel's papers on the linearization of holomorphic mappings near fixed points. To be more precise, I first invented this problem myself (as a simplified model of the problem of circle mappings) and solved it by Kolmogorov's method. Only later on, I discovered Siegel's work who had obtained the same result by another method in about 1940.
"We are in a good company," Kolmogorov told me when I let him know of my bibliographic findings. As far as I understand, he was aware of neither Siegel's works nor J. E. Littlewood's[25] works on the exponential slowness of an increase in perturbations.» (Arnol'd 1997, p. 738).

Although Siegel's work undoubtedly contributed to the development of celestial mechanics, the extent to which Kolmogorov may have drawn upon his research remains uncertain. When Siegel published his 1942 article, he was already in the United States, as World War II was redrawing borders and dividing nations. It is quite plausible that the German mathematician's work did not reach Soviet borders for several years, particularly during the Cold War.

In this regard, in a paper from the 1960s, Arnol'd places their names side by side in the opening lines of the introduction:

«The difficulty of qualitative problems of classical mechanics is well known. In spite of prolonged efforts by many mathematicians most of these problems still await solution. Only in recent times, beginning with the work of C.L. Siegel (1942) and A.N. Kolmogorov (1954), has some progress been made in solving problems on the stability of motion of dynamical systems.» (Arnol'd 2009, p 307).

In a more recent account, Arnol'd returned to the question with additional details – including a critique of academic practices in the United States:

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[24] Arnaud Denjoy (1884-1974), French mathematician known for his work on real-variable functions.

[25] John Edensor Littlewood (1885-1977), English mathematician best known for his contributions to function theory and series theory, many of which were developed in collaboration with Godfrey Harold Hardy.

«Just at this time Kolmogorov was giving a course at Moscow University on his work on small denominators and on Hamiltonian systems and on what is now called KAM theory. [...] I came to Kolmogorov with my theorems.
"Well," he said, "here is my paper in Doklady '54. I think it will be good if you continue with this problem, try to think of applications to celestial mechanics and rigid body rotation. I am very glad that you have chosen a good problem." [...]
I read other people's works and I finally discovered some papers by Siegel, who was a personal friend of Kolmogorov when they stayed in Göttingen in the 1930s.[26] Kolmogorov was not aware that Siegel had later worked on the small denominators problem. Siegel's paper was published in 1941[27] but was unknown to Kolmogorov. He knew about the works of Poincaré, of Denjoy, and of Birkhoff, but not about Siegel. So he told me that we were in very good company:
"Siegel is really serious," he said.
I had discovered the Siegel theorem related to the normal forms for circle rotations because of the system of education at Moscow University, which was different from that in America. I think it followed the German tradition that, when you have a result and wish to publish it, you first have to check the literature to see whether someone else has ever studied it. We were told this in our first introductory course in library work, in which we were taught how to find, starting from zero information, everything needed. There was no Internet at that time of course, but still we were able to find the references, and this is how I discovered that Siegel existed.» (Arnol'd 2004, p. 615).

Futhermore, on May 28, 2021, I had the opportunity to interview Yakov Grigorievich Sinai.[28] My first question concerned precisely this issue:

[26] Siegel was in Frankfurt when Kolmogorov visited Göttingen in 1930-31. At that time, Soviet mathematicians were not even allowed to write their papers in languages other than Russian – let alone travel abroad. In (Shiryaev 1989)), where some details of Kolmogorov's trip and the places he visited are reported, neither Siegel nor Frankfurt is mentioned. See also (Kolmogorov 1986). However, it remains possible that they met in the 1930s. Kolmogorov's papers, partially published in 2003 (see *Introductory Note to the Bibliography*), may help trace these kinds of contacts.

[27] Arnol'd is likely referring to Siegel's 1942 article here, although he appears to have the date incorrect.

[28] For the complete interview see (Chierchia, Fascitiello 2024, appendix).

F: The first question concerns Siegel's work on Diophantine estimates. These techniques are also used by Kolmogorov in his proof of the theorem in 1954, but he did not mention Siegel in the bibliography. Do you know if Kolmogorov was aware of Siegel's work on such a matter?

S: In my opinion, he didn't know Siegel's work. Siegel's work was discussed later in Arnol'd's seminar, and I assume that Arnol'd explained Siegel's work to Kolmogorov. As you know, they both used small denominators.

F: Do you know what inspired Kolmogorov for Diophantine estimates?

S: I'm not so sure about this.

As a matter of fact, Kolmogorov never mentioned Siegel's work – neither in the two papers published in Doklady, nor in the extensive bibliography of the published version of the Amsterdam lecture, nor in the note accompanying his work on classical mechanics in Volume 1 of the *Selected Works*.

Finally, it is worth recalling that Moser, who was a student of Sinai, was asked by «Mathematical Reviews» to review the published text of the Amsterdam lecture (Kolmogorov 1957). At that time, he had already been working on the stability of elliptic fixed points of area-preserving mappings, encouraged by Siegel (see Moser 1999). It seems rather surprising that in the review he wrote, there is no mention of any connection with Siegel's work.

This issue presents an interesting case study from a historiographical perspective – touching on both the transmission of information in the scholarly world and the evolution of ideas in mathematics – and it deserves further investigation. Whether or not Kolmogorov was aware of Siegel's work does not, in any case, diminish the relevance, impact, or distinctive character of his own contribution.

3.3 The roots of Kolmogorov-Arnol'd-Moser (KAM) Theory

> Of his predecessors the most significant references are to Hadamard, Birkhoff, Borel, Brouwer, Hilbert, Carathéodory, Lebesgue, Luzin, Taylor, von Kármán, Hardy, and Hausdorff, in other volumes to Chebyshev, Bernstein, von Mises, and Fisher. I think that it is precisely them we must number among those of his predecessors to whom he turned most of all. Somewhat surprising is the absence of references to Poincaré. This is largely because Kolmogorov learnt of Poincaré's ideas by reading the works of Chazy and Charlier. The other mathematicians to whom Kolmogorov refers were part of the current scientific scene. Here we must mention the great influence which the works of Krylov-Bogolyubov and de Rham[29] had on him.
>
> (Tikhomirov 1988, p. 23)

Let us now revisit the points in Kolmogorov's Amsterdam lecture where the influence of the mathematical landscape described in Chapter 1 appears most clearly. We will then briefly outline the future directions of his research program.

The work of Bogolyubov and Krylov in the field of nonlinear mechanics appears in Section 4 of the Amsterdam lecture, where Kolmogorov begins to reflect on dynamical systems defined on noncompact manifolds. The study of such systems is made possible, in part, by the extension of ergodic theory introduced in (Krylov, Bogolyubov 1937). In that work, by constructing invariant and transitive measures even in cases where they are not naturally present, they broadened the scope of applicability of ergodic theory.

On this subject, Kolmogorov offers only "some remarks" – as the title of the section reads – without reaching a definitive conclusion, as he does in the compact case (Section 3) regarding the two questions on transitivity and the spectrum. Instead, he outlines a few possible scenarios:

[29] Georges de Rham (1903-1990), Swiss mathematician known for his contributions to differential and algebraic topology.

«The arguments which, in the case of a compact Ω, can be given in favour of the idea that a compact dynamical system of "general type" is transitive, when applied to non-compact dynamical systems, leads to the hypothesis that "in general" one of the following two cases holds: either the system is dissipative (that is, almost all its points recede) or the measure *m* is ergodic (obviously, in the latter case the receding points constitute only a set of measure zero).

[...] When it is known in advance that there is a set of positive measure consisting of receding points, then in accordance with what has been said, the conjecture arises that the system is dissipative. *Probably Birkhoff's assumption that the three-body problem is dissipative is based on some consideration of this kind.*» (Kolmogorov 1957, p. 369, my emphasis).

Thus, the study of noncompact manifolds becomes particularly compelling when one considers its implications for celestial mechanics – not only in the three-body problem, as Kolmogorov emphasizes, but also in cases involving the capture or recession of a celestial body. Regarding the latter, he notes that very few scholars are currently drawn to these topics.

As an example of the stagnation in addressing open problems in celestial mechanics, Kolmogorov refers to his own disapproval of certain considerations made in the 1930s by the astronomer Chazy (see above, §1.1), particularly regarding the conjecture that capture is impossible in the three-body problem. He notes that this criticism was only recently confirmed – almost twenty years later – by Schmidt, with further contributions by Sitnikov.

«We note that, among more elementary problems, particular problems dealing with receding trajectories of various specific types attract little attention of specialists in the qualitative theory of differential equations. A spectacular example of this is the fact that a disproof of Chazy's assertions that no "exchange" and "capture" are possible in the three-body problem [17, 18] was first carried out in a way which is cumbersome (and logically unconvincing without precise error estimates), using numerical integration (see Bekker [19] and Shmidt [20]), and only recently has Sitnikov [21] constructed an example of "capture" in a very simple manner and almost without calculations.» (Kolmogorov 1957, p. 370).

In Sections 3 and 4, Kolmogorov addresses the question of the spectrum and its properties for transitive systems, as well as the existence of transitive measures even in cases where the phase space in which the dynamical system is defined is not compact. His interest in measure theory is unsurprising, given his foundational work in this area, particularly his *Foundations of the Theory of Probability* (Kolmogorov 1933).

However, as already mentioned in Section 2.3, Jan von Plato has suggested that Kolmogorov's interest in measure theory may have stemmed from an interest in physics, rather than the other way around.[30]

These issues were also of great interest to von Neumann – along with Birkhoff and Koopman (see above, §1.2). Let us quote Sinai on the connection between research in the 1930s and that of the 1950s, from a 1989 paper on Kolmogorov's work in this area, published in the «Annals of Probability»:

«Apparently the interests of Kolmogorov in ergodic theory had already started in the 1930s. In mathematical Moscow it was a period of construction of the foundations of the theory of stationary random processes. One might recall the paper by Khintchine [he refers to (Khintchine 1938)] at that time dedicated to the spectral theory of such processes. [...]

The paper by Khintchine [he refers to (Khintchine 1933)], where he gave a purely metric proof of the Birkhoff ergodic theorem, belonged to ergodic theory itself. In view of this paper the ergodic theorem on a.e. convergence of time averages is often called the Birkhoff-Khintchine theorem at least in the Russian literature. In the 1930s, the well-known paper by Krylov and Bogolyubov [he refers to (Krylov, Bogolyoubov 1937)] on invariant measures for groups of homeomorphisms of topological spaces was written.

In the beginning of the 1930s, there appeared the famous paper by von Neumann,[31] where the general notion of the metric isomorphism of one-parameter groups of measure-preserving transformations was introduced. Also in [21][32] von

[30] «Two works precede Grundbegriffe's axiomatization of measure theory [Kolmogorov, 1929, 1931]. In the latter, there was a physical motivation for constructing a theory of probability, namely the need to handle schemes of statistical physics in which time and state space are continuous.»

[31] This refers to (von Neumann 1932b); the "fame" of this contribution merits further study.

[32] (von Neumann, J. 1932). "Zur Operatorenmethode in der klassischen Mechanik". «Annals

Neumann proved a basic theorem of metric isomorphism of ergodic dynamical systems with pure point spectrum. This theorem showed that for this class of systems the spectrum is the complete metric invariant. *Since that time the problem of metric classification of dynamical systems became one of the central ones in ergodic theory*. The scientific activity of von Neumann was always under close attention. It is not surprising that this problem became well known quite soon in Moscow and several mathematicians spent a lot of effort trying to make some progress here.

[...] For Kolmogorov the end of the 1930s was the beginning of his classical works on hydrodynamics and turbulence. His first publication which can be considered as relating to ergodic theory goes back to 1937,[33] where he exposed the Birkhoff-Khintchine theorem in probabilistic terms.» (Sinai 1989, p. 833, emphasis mine).

Thus, in Sections 3 and 4 of the Amsterdam lecture, the – somewhat surprising – acknowledgement by Kolmogorov (in the 1985 note on his contributions to classical mechanics, in Volume 1 of his *Selected Works*; see above, Chapter 1) of the influence of von Neumann's writings on the spectral theory of dynamical systems finds full confirmation. Like the American-Hungarian mathematician, Kolmogorov approaches the problem from a broad perspective, seeking general answers through the study of dynamical systems via ergodic and spectral theory.

Further confirmation comes from another part of Sinai's testimony, in which he refers to the already mentioned seminar held in Moscow in the autumn of 1957: Kolmogorov's seminar on dynamical systems.

«In the autumn of 1957, Kolmogorov organized his famous seminar on dynamical systems and gave a lecture course on the same subject. Among the participants and listeners there were Alekseev, Arnol'd, Girsanov, Meshalkin, Pinsker, Sinai, Sitnikov, Tikhomirov and others.

[...] The lectures by Kolmogorov started with the proof of the metric isomorphism of dynamical systems with pure point spectrum. He gave an entirely probabilistic exposition of the corresponding theorem by von Neumann.

[...] In the seminar the participants discussed in much detail the construction of Ito's multiple stochastic integrals and the ergodic properties of Gaussian

of Mathematics», Second Series, 33, 587-642.

[33] This refers to Kolmogorov's paper, in Russian, "On a Simplified Proof of the Birkhoff–Khinchin Ergodic Theorem" (Kolmogorov 1937).

stationary processes. It is well known that such processes can be obtained as natural limits of quasi-periodic processes, that is, processes corresponding to dynamical systems with pure point spectrum. A general feeling at that time was that there exist some boundary separating dynamical systems of probability theory and dynamical systems appearing in ordinary differential equations, classical mechanics and hydrodynamics or, as we call them sometimes, classical dynamical systems.» (Sinai 1989, p. 834).

After the seminar, some of Kolmogorov's students – as he had hoped – devoted themselves to research in mathematical fields connected to his broader program. For example, Sinai published an article on the entropy of dynamical systems at the end of 1957. This work, closely related to ergodic theory and classical mechanics, laid the foundation for further developments in his later papers (Sinai 1959; Sinai 1964).

Arnol'd's early work in the 1960s falls within the framework of Kolmogorov's research program on Hamiltonian systems, with particular relevance to celestial mechanics. As I have shown, Kolmogorov's research in dynamical systems can be traced back to the works of Poincaré and to the unresolved problems in celestial mechanics at the start of the 20th century. However, the key hypothesis of his theorem on the preservation of invariant tori – namely, the non-degeneracy condition (see above, §3.2) – is not satisfied by the system that models the motion of our solar system, and in particular, the three-body problem.

In 1963, Arnol'd published the paper "Proof of a Theorem of A. N. Kolmogorov on the Preservation of Conditionally-Periodic Motions under a Small Perturbation of the Hamiltonian" (Arnol'd 1963a), in which he extended Kolmogorov's results to include certain important cases of degenerate Hamiltonian systems (Arnol'd 1963b; Arnol'd 1964).

Jürgen Kurt Moser – who had reviewed the published version of Kolmogorov's Amsterdam lecture in *Mathematical Reviews* in 1959 (Moser 1959; see below, §3.3.1) – stated.)[34]

I now turn to the proof of Kolmogorov's theorem and its connection to the contributions of Arnol'd and Moser.

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[34] His paper was entitled "On Invariant Curves of Area-Preserving Mappings of an Annulus" (Moser 1962).

3.3.1 KAM theorem or Kolmogorov's theorem?

> The proof of this theorem was published in Dokl. Akad. Nauk SSSR 98 (1954), 527-530, but the convergence discussion does not seem convincing to the reviewer. This very interesting theorem would imply that for an analytic canonical system which is close to an integrable one, all solutions but a set of small measure lie on invariant tori.
>
> (Moser 1959, Mathematical Reviews)

> This theory is called KAM, or Kolmogorov-Arnold-Moser, and people say that there is even a KAM theorem. I was never able to understand what theorem is it.
>
> (Arnol'd 2004, p. 622).

When Moser submitted his review of the 1957 published text of Kolmogorov's lecture to «Mathematical Reviews», he had just turned thirty. In a 1999 recollection of the episode, he wrote that serving as referee for Kolmogorov's lecture – now available in the ICM Proceedings – filled him with enthusiasm, as he had discovered someone else working on Hamiltonian mechanics at a time when interest in the subject was steadily declining.

«Some 40 years ago, when I was at MIT, the Mathematical Reviews asked me to review the famous lecture of Kolmogorov, held at the International Congress 1954 in Amsterdam. This is how I first learned about this work and I was very excited about it. At that time there were few mathematicians interested in Hamiltonian mechanics, and it was encouraging to me *to find others working in this field*. The significance of this fundamental work was indeed apparent to me, since I had been working on the stability problem of elliptic fixed points of area-preserving mappings, a problem C.L. Siegel had urged me to pursue. Naturally, I was disappointed that neither Kolmogorov's address nor his Doklady announcement contained a proof. Therefore I wrote to Kolmogorov asking for the argument. I never received a reply, and I had to write my review not knowing whether this theorem was actually true. I never believed in proof "by authority"! I also had no doubt that Kolmogorov knew how to prove his claims, but that did not help me!» (Moser 1999, p. 19, my emphasis).

Indeed, in the version of the Amsterdam ICM lecture published in the

Proceedings, Kolmogorov states the theorem on the preservation of invariant tori in Section 3 without providing a proof, referring instead to his recently published paper (Kolmogorov 1954) – specifically, to Theorem 1 therein – where the proof must be sought.

In that paper, Kolmogorov outlines – particularly in the central portion of the section devoted to the proof (see above, §3.2) – only the *essential* mathematical steps on which the demonstration of Theorem 1 is based. I have already quoted the theorem's statement; let us now consider the following lines from the three-part argument that follows:

«The transformation

$$(Q, P) = K_\theta(q, p)$$

whose existence in asserted in Theorem 1 can be constructed as the limit of transformations

$$(Q^{(k)}, P^{(k)}) = K_\theta^{(k)}(q, p),$$

where the transformations

$$(Q^{(1)}, P^{(1)}) = L^{(1)}(q, p), \qquad (Q^{(k+1)}, P^{(k+1)}) = L_\theta^{(k+1)}(Q^{(k)}, P^{(k)})$$

are found by a "generalized Newton's method." In this paper, we confine ourselves to the construction of the transformation $K^{(1)} = L^{(1)}$, which makes it possible to understand the role of conditions (3) and (4) of Theorem 1.»[35] (Kolmogorov 1954, pp. 350-351).

One of the distinctive features of Kolmogorov's new approach is the construction of an iterative algorithm that converges very rapidly, inspired by Newton's tangent method for solving algebraic equations. This rapid convergence, in fact, makes it possible to neutralize the influence of small denominators.

Luigi Chierchia has noted that the proof of the theorem is based on two key steps (Chierchia 2008):

[35] Kolmogorov's references correspond to the two conditions mentioned above (§3.2), namely conditions (9) and (10).

1. the construction of successive transformations of variables (q, p), obtained each from the previous one through Newton's quadratic approximation method;
2. the convergence of the iteration process.

Chierchia explains how to construct the Step 1 iterative process, but provides limited detail regarding Step 2 – namely, establishing the convergence of the iteration. Specifically, with respect to the Diophantine condition (condition (3) in his paper; condition (8) above), he identifies only the "somewhat" more difficult part of what would constitute a complete proof.

«Only the applications of condition (3) in the proof of the convergence of the mapping $K_{\theta}^{(k)}$ to an analytic limit mapping K_{Θ} as somewhat more intricate.» (Kolmogorov 1954, p. 352).

Therefore, he nearly takes for granted a crucial element in proving the theorem: the analytic convergence of the sequence of functions $K_{(k)}$ to a function K_{θ}, relying solely on the Diophantine condition on the frequencies as sufficient.

A detailed proof of the theorem – one that could be considered rigorous, fully convincing, or satisfactory – would require further arguments in this regard. For example, it might involve the introduction of a decreasing sequence of Banach spaces (of progressively smaller dimensions), within which the convergence of each function $K_{\Theta}^{(k)}$ can be ensured at every step.

A complete demonstration, which also attempts to retrace the original formulation, has been published by Chierchia, who identifies the "missing parts" in Kolmogorov's original argument:

«We point out that step (ii)[36] – which consists in introducing a scale of Banach spaces, giving recursive estimates and deducing from such estimates the convergence of the scheme – is based on very classical tools (such as Cauchy estimates for analytic functions or the classical Implicit Function Theorem) *obviously well known to Kolmogorov*.» (Chierchia 2008, p. 130, emphasis mine).

[36] As stated above, this refers to the convergence of the product iteration process.

It is possible that the absence of certain details in the original demonstration, combined with Moser's review, contributed to the lingering uncertainty surrounding the attribution of the theorem on the persistence of invariant tori in nearly integrable Hamiltonian dynamical systems to Kolmogorov.

«Occasionally, disagreement erupts over how much Kolmogorov proved in 1954 (some say his sketch-of-a-proof had such big gaps that it wasn't a proof at all; others say that it was complete enough to drop the A and M and simply call the KAM theorem 'Kolmogorov's theorem').» (Dumas 2014, p. 15).

In fact, secondary sources and popular accounts (see, for example, Diacu and Holmes 1996; Charpentier, Lesne, and Nikolski 2004) often recount the story of a theorem stated by Kolmogorov but proved almost a decade later by Arnol'd and Moser (though the latter offered a different version). This convergence of the three results – Kolmogorov 1954, Moser 1962, and Arnol'd 1963 – is sometimes collectively referred to as the KAM Theorem.

What remains open to discussion is whether Kolmogorov truly took the missing aspects for granted – perhaps judging them trivial or unimportant – or whether he simply overlooked the demonstrative gap. One may also ask why, when Moser raised doubts about the result and wrote to him seeking clarification, Kolmogorov did not provide the necessary evidence to dispel the uncertainty and definitively validate his theorem. A possible explanation is offered by Arnol'd:

«I turn now to KAM theory. This theory is called KAM, or Kolmogorov-Arnold-Moser, and people say that there is even a KAM theorem. I was never able to understand what theorem is it. In 1954 Kolmogorov proved his marvellous theorem on the preservation of tori in Hamiltonian systems for the case when the Hamiltonian is almost integrable and all functions are analytic. What I contributed was the study of some degenerate cases – when one of the frequencies is zero in the nonperturbed system or when the vicinity of the fixed points or periodic points or tori is of a smaller dimension – and then applications to celestial mechanics. All these cases are separate theorems. My main contribution was the discovery (in 1964) of the universal mechanism of instability in systems with many degrees of freedom, close to integrable (later called "Arnol'd diffusion" by physicists).

In 1962 Moser extended Kolmogorov's theorem to the case of smooth functions. In the first papers of Moser the number of derivatives was enormous. Now we know that in the simplest case of plane rotation you only need three derivatives, and this is just the limit, the critical number of derivatives. But in the beginning the number was 333. For Kolmogorov, this was like a complete change of philosophy (he told me) because he expected, and even claimed in his Amsterdam talk, that the result would be wrong even in C^∞ and that one would need analyticity or something close to it, something like the Gervais condition.[37]
Moser complained that a proof of the theorem in the case of analytic Hamiltonians was never published by Kolmogorov. *I think that Kolmogorov was reluctant to write down the proof because he had other things to do in his remaining years* of active work – which is a challenge when you are sixty. According to Moser, the first proof was published by Arnol'd. My opinion, however, is that Kolmogorov's theorem was proved by Kolmogorov.» (Arnol'd 2004, pp. 622-623, my emphasis).

Whatever the reasons may have been for Kolmogorov's decision not to publish a more complete or satisfying proof of the theorem on the persistence of invariant tori, one thing is certain: the theorems of the three mathematicians involved in this "dispute" are, in fact, distinct. Arnol'd's theorem incorporates, within its formulation, both Theorems 1 and 2 from Kolmogorov's 1954 paper. This is likely one reason why Theorem 2 has remained so "forgotten" over time as a result separate from Theorem 1 – along with the fact that Kolmogorov himself never provided a proof for it (Chierchia, Fascitiello 2024).

Moreover, the proof techniques employed by the three mathematicians differ significantly. In (Moser 1962), the hypotheses of the stated theorem are not the same as those in Kolmogorov's original paper from 1954. Perhaps it would be more accurate in the future to refer to each of these theorems on Hamiltonian systems by their respective authors, using precise titles – such as "Kolmogorov's theorem on the preservation of invariant tori in nearly integrable Hamiltonian dynamical systems."

What is certain is that all three mathematicians made essential

[37] A condition introduced by the mathematician Maurice-Joseph Gevrey (1884-1957), which defines an intermediate function space between that of smooth functions (i.e., C^∞) and real-analytic functions.

contributions to the development of what is now known as KAM theory, which stands as a foundational framework in the study of stability in Hamiltonian dynamical systems. These methods are especially significant for problems modeled as Hamiltonian systems, where stability is established through the proof of convergence of the Fourier series representing the system's solutions.

Concluding remarks

In the overall scientific production of Andrej N. Kolmogorov, a leader of Russian and Soviet mathematics in the 20th century, the contribution to classical mechanics offers historiographical perspectives of interest for understanding his cultural profile. As in the case of John von Neumann, the distinction that was established over the course of the century between pure mathematics and applied mathematics, and the fact that they brought contributions of great originality and cultural impact in both broad areas, could obscure the presence of a unifying vision of the various sectors of mathematics, as well as an affirmation of the necessary link between mathematical research (with its internal motivations of a conceptual and aesthetic nature) and the drive to investigate reality, in all its manifestations.

Kolmogorov's interest and work in classical mechanics began in the 1930s, in years in which attention to it was declining among mathematicians, while physicists affirmed their theoretical independence from classical mathematical physics thanks to new independent theories. Also von Neumann, while concerned with giving a solid axiomatic mathematical foundation to quantum mechanics, was interested in classical mechanics. For both, however, the frame of reference had evolved: the research of George David Birkhoff indicated the need for a general point of view, that would lead to the theory of dynamical systems, which, on the one hand, extended its field of action beyond mechanics, while, on the other, used newly available mathematical tools. I have reconstructed how Kolmogorov approached and was in some way part of the international micro-community pioneering these studies – today central to the complex of mathematics, in particular regarding a great variety of technological and scientific applications. At the same time, in the 1920s-30s the conceptual and cultural attraction towards the open problems of celestial mechanics still persisted – as the core and the roots of modern science, even if in a renewed theoretical framework. In this regard, I have examined the situation of the studies around 1900, a stage in the history of classical mechanics still little studied. In this case, the community of astronomers also comes into play, which had a great development in the Russian Empire between the monarchy and the Bolshevik regime. Kolmogorov himself linked his studies to the fascination for astronomy brought by one of his readings in adolescence, Camille Flammarion, as part of the legacy of the intellectual environment in which he was educated, between the estate of his maternal grandfather near Yaroslav and Moscow. On the other hand, the possibility has been explored that the gap between the interest in these topics in the period between the world wars and the formulation of an actual, plausible research program in the wake of the works of Henri Poincaré in the aftermath of Stalin's death, may have derived from the violent repression of astronomers in the Soviet Union starting from the arrest of Numerov in 1936.

The role of Kolmogorov in Soviet science is an open issue, on which monographic contributions have been published (especially those by Sergei Demidov, in the context of his research on the Moscow mathematical school between the 19th and 20th centuries) or more specific ones, such as those by Alexander Vucinich (author of a remarkable study on the Soviet Academy of Sciences) in his analysis of the philosophical debate on mathematics in the Soviet era. The interest in classical mechanics appears as part of Kolmogorov's vision of the unity of mathematics in its branches and of the desirable harmonization between (abstract) theory and application/practice in mathematics. His attention to the "old fashioned" celestial mechanics supports the fact that he was close to the 19th century ideal *à la Fourier* of the connection between mathematical analysis and the study of nature.

Let's have a closer look at some specific aspects resulting from my investigation:

1. Kolmogorov in front of Poincaré's "Problème général de la Dynamique"

More than sixty years before the formulation of Kolmogorov theorem on the persistence of invariant tori, Poincaré had defined the general problem of dynamics (see above, §1.1.1) as the study of the quasi-periodic motions of a perturbed system written in Hamiltonian form. He had realized that the problem, written in its general form

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \ldots,$$

where F_0 denotes the Hamiltonian function of the unperturbed system and μ the perturbation parameter, did not concern only aspects of celestial mechanics – so dear to Poincaré – but all those problems of mechanics "close" to integrable problems.

Such an approach could potentially offer developments in many areas of mechanics, by providing knowledge on all those problems which can be connected, through the theory of perturbations, to the few known integrable systems. In fact, while completely integrable systems are very few (see §1.2), those close to integrable systems, in the sense of perturbation theory, are many. Thus, it is easy to understand why Poincaré talked about "general" problem. Kolmogorov provided a crucial contribution, as he opened a path to deal with the problem, that led to subsequent developments: given the condition of non-degeneracy[1] on the unperturbed Hamiltonian system F_0, given a very large set of frequencies in the set of real numbers (meaning that its complement is a set of zero Lebesgue measure), all frequencies satisfying the Diophantine condition[2], and given a sufficiently small enough perturbation μ,[3] most invariant tori present in the integrable unperturbed Hamiltonian system continue to survive. Each torus will deform only slightly with respect to the unperturbed torus having the same frequency and so, in the phase space of the perturbed system, there are equally invariant tori, over which the motions are nearly periodic. Furthermore, in the phase space of the perturbed system, it turns out that the invariant tori are the majority, in the sense that the Lebesgue measure of the complement of their union is small and depends on the perturbation μ of the system.

2. A micro-community working on frontiers of mathematics in the 1930s

If the line connecting Kolmogorov's contribution to some results and conjectures left open by Poincaré appears quite clarified (Diacu, Holmes 1996, Barrow-Green 1997, Dumas 2014), a further, previously invisible thread links Kolmogorov to other contemporary scholars working mainly in the USSR and the USA, forming in a sort of small international scientific community, strongly united in the intentions and research methods used.

The apparent hiatus of more than fifty years between Kolmogorov and

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[1] The second condition in Kolmogorov's theorem on the persistence of invariant tori, above in §3.2. Here, I am using current notation, whereas earlier I used Kolmogorov's original notation.

[2] The first condition in Kolmogorov's theorem on the persistence of invariant tori, above in §3.2.

[3] In Kolmogorov's original 1954 paper, the perturbation corresponds to the parameter θ (see §3.2).

Poincaré is filled by this network which, in the 1930s, connects works stemming from of classical mechanics going even beyond the qualitative/geometrical methods outlined in Poincaré: von Neumann, Birkhoff, Koopman, Krylov and Bogolyubov up to, of course, Kolmogorov.

This mathematical micro-community, although divided geographically (three in the United States, two in Ukraine and one in Russia), thoroughly applied measure theory to assess the bulk of different behaviours in dynamical systems, as well as the theory of operators (thus considering the properties of functions defined on a Hilbert space).

This was the approach used by Koopman, Birkhoff and von Neumann, who saw the evolution of the ergodic theory and the formalizations of the homonymous theorems - by Birkhoff and von Neumann - driven by the theorem obtained by Koopman connecting Hamiltonian systems with unitary operators. And in this wake, von Neumann's interest in a further connection emerges: the spectral theory, which connects a dynamical system to its spectrum, in an attempt – again – to provide information on a dynamical system by deriving it from properties concerned some operators connected to it (the eigenfunctions defined by the eigenvalues of the spectrum, continuous or discrete).

Working in Kiev, Krylov and Bogolyubov took up the work of these three American mathematicians and extended them to more general systems. In fact, in conservative Hamiltonian systems there is a natural measurement function (the conservation of volume, for example) whose existence is a necessary condition for the development of the techniques developed by von Neumann in the field of ergodic theory. Conversely, in nonlinear systems, which often represent a dissipative dynamic, this measure is not present in a natural way. Non-linear mechanics received great attention in the Soviet Union in the 1930s because of its technological potential applications. Thus, in an attempt to apply the same approaches of their overseas colleagues, Krylov and Bogolyubov built a measure function in nonlinear systems, with the same properties as the one existing for Hamiltonian systems, starting from which all the approaches used for the latter could be transferred to the study of non-conservative systems.

This new method, which transforms the study of differential equations or the Hamiltonian in classical mechanics into a study in the field of operator theory and spectral theory, was the key to Kolmogorov's work: his research program perfectly reflects this methodology of research followed by the colleagues just mentioned. Furthermore, the study of the possible transitivity of the system (or ergodicity) and the study of its spectrum (continuous or discrete), allow to answer questions about "general" or "typical" properties of almost integrable perturbed Hamiltonian systems, instead of properties of single, specific systems.

3. A new paradigm in classical mechanics

The contributions of the 1930s and 1950s, including the landmark 1954 paper and Amsterdam lecture by Kolmogorov that I have considered, are rooted in the centenary history of classical mechanics, that has had a paradigmatic role in the evolution of modern science and the extension of its methods beyond the inanimate.

The new mathematical approach and the corresponding formalism, introduced at the beginning of the 20th century, have brought a deep change in the approach to classical mechanics, but – even avoiding any cumulative view on the evolution of science – past contributions (Lagrange, Hamilton, Jacobi, Poincaré etc) are embedded and hidden in modern views. In spite of the useful, successful applications of theoretical research, hard basic problems such as the three body problem and the stability of the Solar system, between theoretical schemas and real phenomena, appear as a fortress difficult to conquer. Continuing this metaphor, mathematical schemas appear as ways to attack ancient problems from different points of the enclosure and with ever more sophisticated weapons. Mathematical analysis developed in symbiosis with mechanical issues, until the turn to the qualitative approach from a more geometric and topological theory conceived by Poincaré at the end of the 19th century.

In the 1921 revision of his work, Poincaré offered a reflection – from his conventionalist point of view – on the interaction between physical objects and theoretical "form" or "clothes":

«Dans les théories physiques, il faut distinguer le fond et la forme. Le fond, c'est l'existence de certains rapports entre des objets inaccessibles. Ces rapports sont la seule réalité que nous puissions atteindre et tout ce que nous pouvons demander, c'est qu'il y ait les mêmes rapports entre ces objets réels inconnus et les images que nous mettons à leur place. La forme n'est qu'une sorte de vêtement dont nous habillons ce squelette; ce vêtement, nous le changeons fréquemment, à l'étonnement des gens du monde, que cette instabilité fait sourire et qui proclament la faillite de la

Science. Mais si la forme change souvent, le fond reste.
Les hypothèses relatives à ce que je viens d'appeler la forme ne peuvent pas être vraies ou fausses, elles ne peuvent être que commodes ou incommodes. Par exemple, l'existence de l'éther, celle même des objets extérieurs ne sont que des hypothèses commodes. C'est pour cela que l'on voit renaître de leurs cendres en se transformant certaines théories que l'on croyait définitivement abandonnées. C'est pour cela aussi qu'il y a certaines catégories de faits qui s'expliquent également bien dans deux ou plusieurs théories différentes, sans qu'aucune expérience puisse jamais décider. Cela est vrai en particulier pour les théories mécanistes. On peut en effet démontrer que, si un phénomène comporte une explication mécanique, il en comportera une infinité.» (Poincaré 1921, p. 130).

Kolmogorov's effort is a nice example of the new general visions suggested by developments in mathematics, and at the same time the search for a new space and reconfiguration for classical mechanics after the birth of theoretical physics. The aspects we have considered, suggested by the exploration of the roots of Kolmogorov's contribution, show its persistence in modern mathematical thought, despite the sharp divisions between pure and applied mathematics. As I recalled in the introduction, Dumas has pointed out the need to build a narrative to understand the meaning of the development of KAM theory in 20th century science, hidden also because of the split of the mathematical international community during the central decades of the century. Reconstructing the roots of Kolmogorov's contribution has included human and almost biographical roots, cultural roots, conceptual roots, thus showing the relevance of historiographical endeavor to enlarge and deepen our understanding of the role of science in contemporary world.

Bibliography

Introductory note

Three volumes of *Selected Works* by Andrej N. Kolmogorov were published in 1985, 1986, and 1987 by Nauka (Moscow), just before he passed away. The editors were Vladimir Mikhailovich Tikhomirov (vol. 1) and Albert Nikolaevich Shiryaev (vols 2-3). The full English translation of the three volumes was published in 1991, 1992, and 1993 by Kluwer Academic Publishers (Dordrecht) (Kolmogorov, 1991-1993). The translators were three mathematicians: Vladimir M. Volosov (vol. 1, including the contributions to mechanics)[1], Anders Gunnar Lindquist (vol. 2), and Alexei Sossinski (vol. 3).

A complete list of Kolmogorov's scientific output was published in 1989 in «The Annals of Probability» by the Institute of Mathematical Statistics (17(3), pp. 945-964, section "Memorial Articles").

In 1988, the journal «Uspekhi Matematicheskikh Nauk» devoted its final issue (no. 6, December) to the commemoration of Kolmogorov, with 11 contributions, including the biographical essay by Tikhomirov (1988).

In 1990, a 70-page obituary-study edited by David Kendall was published by the London Mathematical Society, including 11 contributions (Kendall 1990); for KAM Theory, see Moffatt (1990).

In 2000, the American Mathematical Society published a volume entitled *Kolmogorov in Perspective* (History of Mathematics, vol. 20, Providence, R.I.: AMS, 2000, 8 contributions).

A second collective volume, *L'héritage de Kolmogorov en mathématiques*, was published by Belin (Paris) in 2004 and translated into English in 2007. It includes fifteen contributions (Charpentier *et al.*, 2004); for classical mechanics, see Ghys (2004) and Hubard (2004).

The twin volume, *L'héritage de Kolmogorov en physique*, by the same French

[1] Vladimir Markovich Volosov (b. 1928) pursued his scientific career in the field of nonlinear mechanics, with a focus on ordinary and partial differential equations, at Lomonosov Moscow State University. He graduated in 1950 from the Faculty of Physics, Department of Mathematics, earned his Ph.D. in 1956, and his D.Sc. in 1961. He later became a professor and a member of the P. P. Shirshov Institute of Oceanology of the Russian Academy of Sciences. Volosov also translated a 1980 textbook by Viacheslav M. Starzhinskii (1918-1993), which was published in English by MIR (Moscow) under the title *Advanced Course of Theoretical Mechanics for Engineering Studies* (1982). See <www.mathnet.ru/eng/person12469>.

publisher (Livi & Vulpiani, eds, 2003), was translated into English in 2010. Part I is devoted to Chaos and dynamical systems (Livi, Ruffo, Shepelyansky, 2003; Celletti, Froeschlé, Lega, 2003).

In 2003, an international conference was held in Moscow to commemorate the centenary of his birth, titled "Kolmogorov and Contemporary Mathematics" (Shirayev 2004); the proceedings were published in Moscow (editors: Urij Sergeevich Osipov, Viktor Antonovich Sadovnichij, and Albert Nikolaevich Shiryaev).

Some of Kolmogorov's letters to Arnol'd were published in Arnol'd (2000). In 2002, the Moscow publisher Fizmatlit published Kolmogorov (3 vols., edited by Shiryaev, with contributions from Natal'ya Grigor'evna Khimchenko [1937-2007]), including excerpts from the Kolmogorov-Alexandrov correspondence (vol. 2) and Kolmogorov's private diaries (vol. 3). See Duzhin (2011) for the English translation of some 1943 annotations, and Khimchenko (2001).

The website www.kolmogorov.com includes a wide range of materials on Kolmogorov.

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The powerful methods of modern mathematics since the seventeenth century were first tested in the field of mechanics. What became of classical methods after the advent of twentieth-century theoretical physics? This volume examines Russian scholar Andrej N. Kolmogorov's contribution to classical mechanics, offering a reconstruction of the origins of the research program that he made public, just over a year after Stalin's death, in a lecture at the Amsterdam International Conference of Mathematicians in 1954. The author adopts a cultural history of mathematics perspective to offer a historical narrative that weaves together mathematical open problems, the Soviet intellectual context, the complex international network of scholars involved, and Kolmogorov's intellectual biography. The book is aimed at historians of mathematics and science, and scholars interested in the evolution of scientific thought in the twentieth century.

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